[Name]
[Section #]
Homework #5
Math 527

1

The equation to solve is

$$\left(D^2+4\right)y=\sin(x)\,. \tag{1.1}$$

Solve the associated homogeneous equation.

$$\begin{aligned}\left(D^2+4\right)y_* &= 0\\ (D-2i)(D-2i)\,y_* &= 0\\ y_* &= C_1\cos(2x)+C_2\sin(2x)\end{aligned} \tag{1.2}$$

Now make the **judicious guess**

$$y=\alpha\cos(x)+\beta\sin(x)\,. \tag{1.3}$$

Then

$$\begin{aligned}\left(D^2+4\right)y &= \left(-\alpha\cos(x)-\beta\sin(x)\right)+4\left(\alpha\cos(x)+\beta\sin(x)\right)\\ &= 3\alpha\cos(x)+3\beta\sin(x)\,;\end{aligned}$$

substituting this expression into (1.1) gives

$$3\alpha\cos(x)+3\beta\sin(x)=\sin(x)\,,$$

which implies

$$\alpha=0 \text{ and } \beta=\frac{1}{3}.$$

Substitute these values into (1.3).

$$y=\frac{1}{3}\sin(x)$$

The sum of this solution with (1.2) is the general solution.

$$y=\frac{1}{3}\sin(x)+C_1\cos(2x)+C_2\sin(2x)$$

2

The equation to solve is

$$\left(D^2+4\right)y=\sin(2x). \tag{2.1}$$

We already have the homogeneous solution in (1.2).

The **judicious guess** $y=\alpha\cos(2x)+\beta\sin(2x)$ won't work, since it's already a homogeneous solution. So instead use

$$y=x\left(\alpha\cos(2x)+\beta\sin(2x)\right). \tag{2.2}$$

Then, by the product rule,

$$\begin{aligned} D^2y &= 2\left(-2\alpha\sin(2x)+2\beta\cos(2x)\right)+\left(-4\alpha\cos(2x)-4\beta\sin(2x)\right) \\ &= 4\beta\cos(2x)-4\alpha\sin(2x)-4x\left(\alpha\cos(2x)+\beta\cos(2x)\right), \end{aligned}$$

so

$$\left(D^2+4\right)y=4\beta\cos(2x)-4\alpha\sin(2x).$$

Substituting this expression into (2.1) gives

$$4\beta\cos(2x)-4\alpha\sin(2x)=\sin(2x).$$

Solve for α and β.

$$\begin{array}{ccc} 4\beta=0 & \text{and} & -4\alpha=1 \\ \beta=0 & \text{and} & \alpha=-\frac{1}{4} \end{array}$$

Substitute these values into (2.2).

$$y=-\frac{1}{4}x\cos(2x)$$

The sum of this solution with (1.2) is the general solution.

$$y=-\frac{1}{4}x\cos(2x)+C_1\cos(2x)+C_2\sin(2x)$$

3

The equation to solve is

$$\left(D^2 - 4D + 10\right) y = e^{-t} \sin(t). \tag{3.1}$$

Solve the associated homogeneous equation.

$$\left(D^2 - 4D + 10\right) y_* \quad = \quad 0$$

Solve the characteristic equation.

$$\begin{aligned} \lambda^2 - 4\lambda + 10 &= 0 \\ \lambda &= \frac{4 \pm \sqrt{16 - 40}}{2} \\ &= 2 \pm i\sqrt{6} \end{aligned}$$

These are two nonreal roots. So,

$$y_* = e^{2t}\left(C_1 \cos\left(\sqrt{6}t\right) + C_2 \sin\left(\sqrt{6}t\right)\right). \tag{3.2}$$

Now, make the **judicious guess**

$$y = e^{-t}\left(\alpha \cos(t) + \beta \sin(t)\right). \tag{3.3}$$

For convenience, substitute $s = \alpha \cos(t) + \beta \sin(t)$. Then

$$\begin{aligned} y &= e^{-t} s; \\ Dy &= e^{-t}\left(s' - s\right); \\ D^2 y &= e^{-t}\left(s'' - 2s' + s\right). \end{aligned}$$

Combining these gives

$$\begin{aligned} \left(D^2 - 4D + 10\right) y &= e^{-t}\left(\left(s'' - 2s' + s\right) - 4\left(s' - s\right) + 10s\right) \\ &= e^{-t}\left(s'' - 6s' + 15s\right). \end{aligned}$$

Note that

$$\begin{aligned} 15s &= 15\alpha \cos(t) &+& \quad 15\beta \sin(t); \\ -6s' &= -6\beta \cos(t) &+& \quad 6\alpha \sin(t); \\ s'' &= -\alpha \cos(t) &+& \quad -\beta \sin(t); \end{aligned}$$

and therefore

$$\left(D^2 - 4D + 10\right) y = e^{-t}\left(\left(14\alpha - 6\beta\right) \cos(t) + \left(6\alpha + 14\beta\right) \sin(t)\right).$$

Substitute this expression into (3.1).

$$e^{-t}\left(\left(14\alpha-6\beta\right)\cos(t)+\left(6\alpha+14\beta\right)\sin(t)\right)=e^{-t}\sin(t)$$

Solve for α and β.

$$\begin{array}{rcl}14\alpha-6\beta & = & 0 \\ \alpha & = & \frac{6}{14}\beta \\ \vdots & \vdots & \vdots \\ \alpha & = & \frac{6}{14}\cdot\frac{7}{116} \\ & = & \frac{3}{116}\end{array}\quad\begin{array}{c}\text{and}\\ \text{and}\\ \vdots \\ \text{and}\end{array}\quad\begin{array}{rcl}6\alpha+14\beta & = & 1 \\ 6\left(\frac{6}{14}\beta\right)+14\beta & = & 1 \\ \frac{116}{7}\beta & = & 1 \\ \beta & = & \frac{7}{116}\end{array}$$

Substitute these values into (3.3).

$$y=e^{-t}\left(\frac{3}{116}\cos(t)+\frac{7}{116}\sin(t)\right)$$

The sum of this solution with (3.2) is the general solution.

$$y=e^{-t}\left(\frac{3}{116}\cos(t)+\frac{7}{116}\sin(t)\right)+e^{2t}\left(C_1\cos\left(\sqrt{6}t\right)+C_2\sin\left(\sqrt{6}t\right)\right)$$

4

The equation to solve is

$$\left(D^3 - 2D^2 - 4D + 8\right) y = 6xe^{2x}. \tag{4.1}$$

Solve the associated homogeneous equation.

$$\begin{aligned}
\left(D^3 - 2D^2 - 4D + 8\right) y_* &= 0 \\
(D-2)^2 (D+2)\, y_* &= 0 \\
y_* &= (C_0 + C_1 x)\, e^{2x} + C_2 e^{-2x}
\end{aligned} \tag{4.2}$$

The obvious **judicious guess**, $y = e^{2x}\left(\alpha x + \beta\right)$, won't work since it's already a homogeneous solution. So instead use

$$y = e^{2x}\left(\alpha x^3 + \beta x^2\right). \tag{4.3}$$

For convenience, substitute $s = \alpha x^3 + \beta x^2$. Then

$$\begin{aligned}
y &= e^{2x} s; \\
Dy &= e^{2x}\left(s' + 2s\right); \\
D^2 y &= e^{2x}\left(s'' + 4s' + 4s\right); \\
D^3 y &= e^{2x}\left(s''' + 6s'' + 12s' + 8s\right).
\end{aligned}$$

Combining these gives

$$\begin{aligned}
\left(D^3 - 2D^2 - 4D + 8\right) y &= e^{2x}\left(\left(s''' + 6s'' + 12s' + 8s\right) - 2\left(s'' + 4s' + 4s\right) - 4\left(s' + 2s\right) + 8s\right) \\
&= \left(s''' - 4s''\right) e^{2x}.
\end{aligned}$$

Note that

$$\begin{aligned}
s'' &= 6\alpha x + 2\beta; \\
s''' &= 6\alpha;
\end{aligned}$$

and therefore

$$\begin{aligned}
\left(D^3 - 2D^2 - 4D + 8\right) y &= e^{2x}\left(6\alpha + 24\alpha x + 8\beta\right) \\
&= 24\alpha x e^{2x} + \left(6\alpha + 8\beta\right) e^{2x}.
\end{aligned}$$

Substitute this expression into (4.1).

$$24\alpha x e^{2x} + \left(6\alpha + 8\beta\right) e^{2x} = 6xe^{2x}$$

Solve for α and β.

$$\begin{aligned} 24\alpha = 6 \quad &\text{and} \quad 6\alpha + 8\beta = 0 \\ \alpha = \tfrac{1}{4} \quad &\text{and} \quad \tfrac{3}{2} + 8\beta = 0 \\ & \qquad \beta = -\tfrac{3}{16} \end{aligned}$$

Substitute these values into (4.3).

$$y = e^{2x}\left(\frac{1}{4}x^3 - \frac{3}{16}x^2\right)$$

The sum of this solution with (4.1) is the general solution.

$$y = \left(C_0 + C_1 x - \frac{3}{16}x^2 + \frac{1}{4}x^3\right)e^{2x} + C_2 e^{-2x}$$

5

The equation to solve is

$$\left(D^2+2D+1\right)y=e^{-t}\ln(t).$$

Solve the associated homogeneous equation.

$$\begin{aligned}\left(D^2+2D+1\right)y_* &= 0\\ (D+1)^2 y_* &= 0\\ y_* &= C_1 e^{-t}+C_2 te^{-t}\end{aligned}$$

Now, use **variation of parameters**. The general solution will be

$$y=u_1e^{-t}+u_2te^{-t}$$

with u_1', u_2' such that

$$\begin{aligned} e^{-t}u_1'+te^{-t}u_2' &= 0;\\ -e^{-t}u_1'+(1-t)e^{-t}u_2' &= e^{-t}\ln(t).\end{aligned}$$

Divide both equations through by e^{-t}:

$$\begin{aligned} u_1'+tu_2' &= 0; && (5.1)\\ -u_1'+(1-t)u_2' &= \ln(t). && (5.2)\end{aligned}$$

To isolate u_2, add (5.1) and (5.2).

$$\begin{aligned} u_2' &= \ln(t)\\ u_2 &= \int \ln(t)\\ &= t(\ln(t)-1)+C_2\end{aligned}$$

To get u_1, substitute u_2' into (5.1).

$$\begin{aligned} u_1'+t\ln(t) &= 0\\ u_1' &= -t\ln(t)\\ u_1 &= -\int t\ln(t)\\ &= -t^2\left(\frac{1}{2}\ln(t)-\frac{1}{4}\right)+C_1.\end{aligned}$$

Therefore the general solution is

$$y = \left(-t^2\left(\frac{1}{2}\ln(t) - \frac{1}{4}\right) + C_1\right)e^{-t} + (t(\ln(t) - 1) + C_2)\,te^{-t},$$

that is,

$$y = \frac{1}{2}t^2e^{-t}\ln(t) - \frac{3}{4}t^2e^{-t} + C_1e^{-t} + C_2te^{-t}.$$

6

The equation to solve is

$$\left(D^2 - 2D + 2\right) y = e^x \sec(x).$$

Solve the associated homogeneous equation.

$$\begin{aligned}
\left(D^2 - 2D + 2\right) y_* &= 0 \\
(D - (1 - i))(D - (1 + i))\, y_* &= 0 \\
y_* &= C_1 e^x \cos(x) + C_2 e^x \sin(x)
\end{aligned}$$

Now, use **variation of parameters**. The general solution will be

$$y = u_1 e^x \cos(x) + u_2 e^x \sin(x),$$

where

$$\begin{aligned}
u_1' &= \frac{-y_1 f(x)}{y_1 y_2' - y_2 y_1'} \\
&= \frac{-\left[e^x \sin(x)\right]\left[e^x \sec(x)\right]}{\left[e^x \cos(x)\right] \frac{d}{dx}\left[e^x \sin(x)\right] - \left[e^x \sin(x)\right] \frac{d}{dx}\left[e^x \cos(x)\right]} \\
&= -\frac{e^{2x} \tan(x)}{e^{2x} \cos(x)(\sin(x) + \cos(x)) - e^{2x} \sin(x)(\cos(x) - \sin(x))} \\
&= -\frac{\tan(x)}{\sin(x)^2 + \cos(x)^2} \\
&= -\tan(x); \\
u_1 &= \int -\tan(x)\, dx \\
&= -\ln(\cos(x)) + C_1
\end{aligned}$$

and

$$\begin{aligned}
u_2' &= \frac{y_2 f(x)}{y_1 y_2' - y_2 y_1'} \\
&= \frac{\left[e^x \cos(x)\right]\left[e^x \sec(x)\right]}{\left[e^x \cos(x)\right] \frac{d}{dx}\left[e^x \sin(x)\right] - \left[e^x \sin(x)\right] \frac{d}{dx}\left[e^x \cos(x)\right]} \\
&= \frac{e^{2x}}{e^{2x}} \\
&= 1; \\
u_2 &= \int dx \\
&= x + C_2.
\end{aligned}$$

Therefore the general solution is

$$y = (\ln(\cos(x)) + C_1)\, e^x \cos(x) + (x + C_2)\, e^x \sin(x),$$

that is,

$$y = e^x (\cos(x) \ln(\cos(x)) + x \sin(x) + C_1 \cos(x) + C_2 \sin(x)).$$

7

The equation to solve is equivalent to

$$\left(D^2+\frac{k}{m}\right)y \quad = \quad \frac{F}{m}\sin\left(\sqrt{\frac{k}{m}}t\right).$$

Make the substitutions

$$w=\sqrt{\frac{k}{m}};\; a=\frac{F}{m} \tag{7.1}$$

to simplify to

$$\left(D^2+w^2\right)y=a\cdot\sin(wt). \tag{7.2}$$

Solve the associated homogeneous equation.

$$\begin{aligned}\left(D^2+w^2\right)y_* &= 0\\ (D-iw)(D+iw)\,y_* &= 0\\ y_* &= C_1\cos(wt)+C_2\sin(wt) \qquad (7.3)\end{aligned}$$

Now, make the judicious guess

$$y=t\left(\alpha\cos(wt)+\beta\sin(wt)\right). \tag{7.4}$$

Then

$$\begin{aligned}D^2y &= 2\left(-w\alpha\sin(wt)+w\beta\cos(wt)\right)+t\left(-w^2\alpha\cos(wt)-w^2\beta\sin(wt)\right)\\ &= 2w\beta\cos(wt)-2w\alpha\cos(wt)-w^2t\left(\alpha\cos(wt)+\beta\sin(wt)\right),\end{aligned}$$

so

$$\left(D^2+w^2\right)y=2w\beta\cos(wt)-2w\alpha\sin(wt).$$

Substitute this expression into (7.2).

$$2w\beta\cdot\cos(wt)-2w\alpha\cdot\sin(wt)=a\cdot\sin(2t)$$

Solve for α and β.

$$\alpha=-\frac{a}{2w}\text{ and }\beta=0$$

Substitute these values into (7.4).

$$y=-\frac{a}{2w}t\cos(wt) \tag{7.5}$$

The general solution is the sum of (7.5) and (7.3).

$$
\begin{aligned}
y &= -\frac{a}{2w} t \cos(wt) + C_1 \cos(wt) + C_2 \sin(wt) \\
y' &= -\frac{a}{2w} (\cos(wt) - wt \sin(wt)) - wC_2 \sin(wt) + wC_1 \sin(wt)
\end{aligned}
$$

Now, use the initial values $y(0) = 0$, $y'(0) = 0$.

$$
\begin{aligned}
0 \cos(0) + C_1 \cos(0) + C_2 \sin(0) &= 0 \\
\frac{a}{2w} (\cos(0) - 0 \sin(0)) - wC_1 \sin(0) + wC_2 \cos(0) &= 0
\end{aligned}
$$

The former equation reduces to

$$C_1 = 0;$$

the latter to

$$
\begin{aligned}
\frac{a}{2w} - wC_2 &= 0 \\
C_2 &= \frac{a}{2w^2}.
\end{aligned}
$$

Substitute these values back into (7.2).

$$
\begin{aligned}
y &= \frac{a}{2w^2} \sin(wt) - \frac{a}{2w} t \cos(wt) \\
&= \frac{a}{2w^2} (\sin(wt) - wt \cos(wt))
\end{aligned}
$$

Reverse the substitutions from (7.1) for the specific solution.

$$y = \frac{F}{2k} \left(\sin\left(\sqrt{\frac{k}{m}} t \right) - \sqrt{\frac{k}{m}} t \cos\left(\sqrt{\frac{k}{m}} t \right) \right).$$

A plot of this function is on the following page.

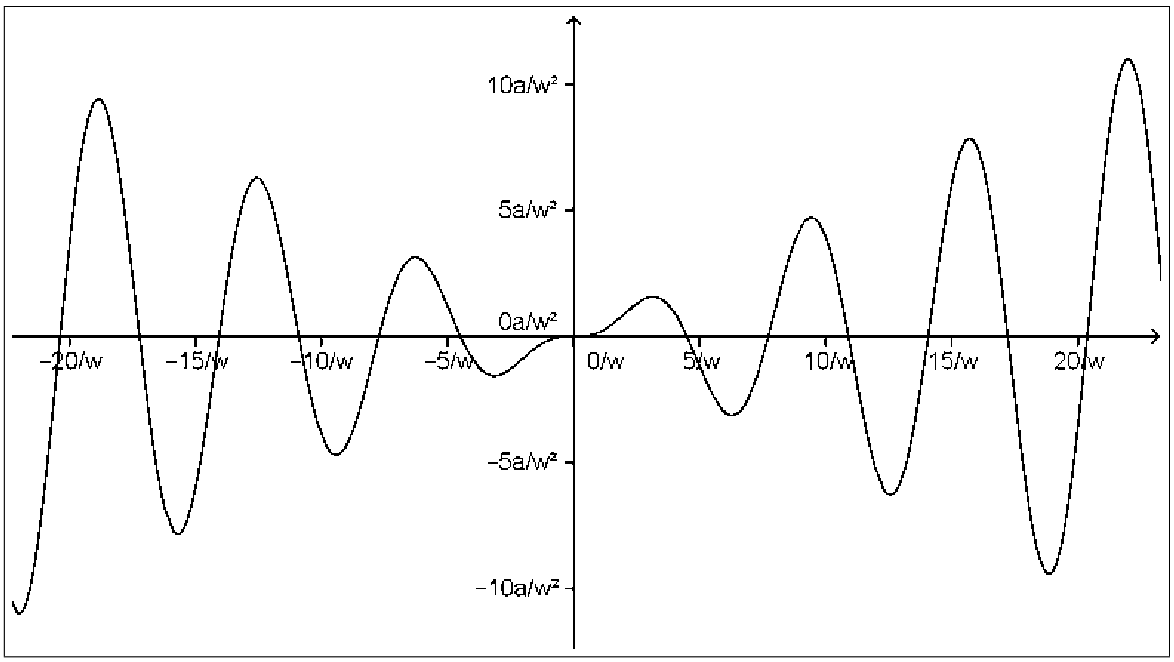

Figure 7.1: Plot of $y = \frac{a}{2w^2} \sin(wt) - \frac{a}{2w} t \cos(wt)$.