Solutions
Section #X
Homework #4
Math 527

1.) $y' - 3y = 0$

ansatz: $y(x) = e^{\lambda x}$ (so $y' = \lambda e^{\lambda x}$)

Then our ODE becomes (after substituting in our ansatz):

$\lambda e^{\lambda x} - 3\lambda e^{\lambda x} = 0.$

Since $e^{\lambda x} \neq 0$, we can divide both sides by $e^{\lambda x}$ to get:

$\lambda - 3 = 0.$

$\lambda = 3.$

So our general solution is $y(x) = c\, e^{3x}$

2.) $y' + 3y = 0$

ansatz: $y(x) = e^{\lambda x}$ (so $y' = \lambda e^{\lambda x}$)

Substituting in yields: $\lambda e^{\lambda x} + 3e^{\lambda x} = 0$

Dividing by $e^{\lambda x}$: $\lambda + 3 = 0$

$\lambda = -3$

So our general solution is $y(x) = ce^{-3x}$

3.) $y'' - 9y = 0$

ansatz: $y(x) = e^{\lambda x}$ (so $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$)

Substituting in yields: $\lambda^2 e^{\lambda x} - 9e^{\lambda x} = 0$

Dividing by $e^{\lambda x}$: $\lambda^2 - 9 = 0$

$(\lambda + 3)(\lambda - 3) = 0$

$\lambda = 3, -3$

So our solutions are $y_1(x) = e^{3x}$, $y_2(x) = e^{-3x}$

So our general solution is $y(x) = c_1 e^{3x} + c_2 e^{-3x}$

4.) $y'' + 9y = 0$

ansatz: $y = e^{\lambda x}$ (so $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$)

Substitution yields: $\lambda^2 e^{\lambda x} + 9e^{\lambda x} = 0$

Dividing by $e^{\lambda x}$: $\lambda^2 + 9 = 0$

$\lambda^2 = -9$

$\lambda = \pm\sqrt{-9} = \pm\sqrt{9}\sqrt{-1} = \pm 3\sqrt{-1} = \pm 3i$ $(\mu = 0, \omega = 3)$

So our solutions are $y_1(x) = e^{3ix} = \cos(3x) + i(\sin(3x))$

$y_2(x) = e^{-3ix} = \cos(3x) - i(\sin(3x))$

So our general solution is $\hat{y}(x) = \hat{c}_1(\cos(3x) + i(\sin(3x))) + \hat{c}_2(\cos(3x) - i\sin(3x))$

But we can skip straight to: $y(x) = c_1\cos(3x) + c_2\sin(3x)$

5.) $y'' - 5y' + 6y = 0$, $y(0) = 1$, $y'(0) = 1$

ansatz: $y = e^{\lambda x}$ (so $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$)

Substitution yields: $\lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} + 6e^{\lambda x} = 0$

Dividing by $e^{\lambda x}$: $\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2, 3$

So our solutions are $y_1(x) = e^{2x}$, $y_2(x) = e^{3x}$

So our general solution is $y(x) = c_1 e^{2x} + c_2 e^{3x}$

Plugging in $y(0) = c_1 e^0 + c_2 e^0 = 1$

$c_1 + c_2 = 1$

Note that $y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x}$

Plugging in $y'(0) = 2c_1 e^0 + 3c_2 e^0 = 1$

$2c_1 + 3c_2 = 1$

Solving for c_1 and c_2 yields $c_1 = 2$, $c_2 = -1$

So our particular solution is $y(x) = 2e^{2x} - e^{3x}$.

6.) $y'' - 6y' + 9 = 0$, $y(0) = 1$, $y'(0) = 1$

ansatz: $y = e^{\lambda x}$ (so $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$)

Substitution yields: $\lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 9e^{\lambda x} = 0$

Dividing by $e^{\lambda x}$: $\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

$\lambda = 3$

Since this is a "double root", our solutions are $y_1(x) = e^{3x}$, $y_2(x) = xe^{3x}$

So our general solution is $y(x) = c_1 e^{3x} + c_2 x e^{3x}$

Plugging in $y(0) = c_1 e^0 + c_2(0)e^0 = 1$

$c_1 = 1$

Note that $y'(x) = 3c_1 e^{3x} + c_2(e^{3x} + 3xe^{3x})$

Plugging in $y'(0) = 3(1)e^0 + c_2(e^0 + 3(0)e^0)$

$1 = 3 + c_2$

$c_2 = -2$

So our particular solution is $y(x) = e^{3x} - 2xe^{3x}$

7.) $y'' + 6y' + 13y = 0$, $y(0) = 1$, $y'(0) = 2$

ansatz: $y = e^{\lambda x}$ (so $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$)

Substitution yields: $\lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 13e^{\lambda x} = 0$

Divide by $e^{\lambda x}$: $\lambda^2 + 6\lambda + 13 = 0$

$\lambda = \frac{-6 \pm \sqrt{36 - 52}}{2} = -3 \pm \frac{1}{2}\sqrt{-16} = -3 \pm 2i$

So our general solution is $y(x) = c_1 e^{-3x}\cos(2x) + c_2 e^{-3x}\sin(2x)$

Plugging in $y(0) = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = 1$

$c_1 = 1$

Notice that $y'(x) = e^{-3x}(c_2(2\cos(2x) - 3\sin(2x)) - c_1(2\sin(2x) + 3\cos(2x)))$

Plugging in $y'(0) = e^0(c_2(2\cos(0) - 3\sin(0)) - (1)(2\sin(0) + 3\cos(0))) = 2$

$2c_2 - 3 = 2 \Rightarrow c_2 = 5/2$

So our particular solution is $y(x) = e^{-3x}[\cos(2x) + 5/2 \sin(2x)]$

8.) Plug in $x = i\omega t$ (where $i = \sqrt{-1}$) into the Taylor series expansion of e^x to show that $e^{i\omega t} = \cos(\omega t) + i(\sin(\omega t))$

Recall: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$

So $e^{i\omega t} = 1 + (i\omega t) + \frac{1}{2}(i\omega t)^2 + \frac{1}{3!}(i\omega t)^3 + \frac{1}{4!}(i\omega t)^4 + \ldots$

$= 1 + i(\omega t) - \frac{1}{2}(\omega t)^2 - \frac{1}{3!}i(\omega t)^3 + \frac{1}{4!}(\omega t)^4 + \ldots$

$= (1 - \frac{1}{2!}(\omega t)^2 + \frac{1}{4!}(\omega t)^4 - \ldots) + i((\omega t) - \frac{1}{3!}(\omega t)^3 + \frac{1}{5!}(\omega t)^5 - \ldots)$

$= \cos(\omega t) + i(\sin(\omega t))$.