[Name]

[Section #]

Homework #3

Math 527

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1. Let $t = 0$ at 8:00 am. Then the information given is:

$$\begin{aligned} \frac{dT}{dt} &= k(T-70); \\ T(0) &= 200; \\ T(2) &= 190. \end{aligned}$$

(a) We'll just solve the system of equations, finding k in the process.

$$\begin{aligned} \frac{1}{T-70}\frac{dT}{dt} &= k \\ \int \frac{dT}{T-70} &= k\int dt \\ \ln|T-70| &= kt + C_0 \\ |T-70| &= e^{kt+C_0} \\ T-70 &= Ce^{kt} \\ T &= 70 + Ce^{kt} \end{aligned}$$

Solve for C, with $t = 0$ and $T = 200$.

$$\begin{aligned} 70 + Ce^{k\cdot 0} &= 200 \\ 70 + C &= 200 \\ C &= 130 \end{aligned}$$

Therefore

$$T = 70 + 130e^{kt}.$$

Solve for k, with $t = 2$ and $T = 190$.

$$\begin{aligned} 70 + 130e^{k\cdot 2} &= 190 \\ 130e^{2k} &= 120 \\ e^{2k} &= \frac{120}{130} \\ &= \frac{12}{13} \\ 2k &= \ln\left(\frac{12}{13}\right) \\ &= \ln(12) - \ln(13) \\ k &= \frac{\ln(12) - \ln(13)}{2} \end{aligned}$$

(This may also be written as $k = \frac{1}{2}\ln\left(\frac{12}{13}\right)$ or, including units, as $k = \frac{\ln(12)-\ln(13)}{2}\ \text{min}^{-1}$.)

(b) When $T = 175$, this means

$$70 + 130e^{kt} = 175.$$

Solve for t:

$$\begin{aligned}
130e^{kt} &= 105 \\
e^{kt} &= \frac{105}{130} \\
&= \frac{21}{26} \\
kt &= \ln\left(\frac{21}{26}\right) \\
&= \ln(21) - \ln(26) \\
t &= \frac{\ln(21) - \ln(26)}{k}
\end{aligned}$$

Use the known value of k from (a).

$$\begin{aligned}
t &= \frac{2(\ln(21) - \ln(26))}{\ln(12) - \ln(13)} \\
&\approx 5.3
\end{aligned}$$

The coffee will be 175 degrees at 8:05 am.

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2. The information given is:

$$\begin{aligned}
m\frac{dv}{dt} &= mg - kv \\
v(0) &= 0 \\
\frac{ds}{dt} &= v \\
s(0) &= 0
\end{aligned}$$

(a) At terminal velocity, $\frac{dv}{dt} = 0$, so

$$\begin{aligned}
0 &= mg - kv \\
kv &= mg \\
v &= \frac{mg}{k}
\end{aligned}$$

The terminal velocity is $\frac{mg}{k}$.

(b) Solve for v as a function of t.

$$\begin{aligned}
m\frac{dv}{dt} &= -k\left(v-\frac{mg}{k}\right)\\
\frac{dv}{dt} &= -\frac{k}{m}\left(v-\frac{mg}{k}\right)\\
\frac{1}{v-\frac{mg}{k}}\frac{dv}{dt} &= -\frac{k}{m}\\
\int\frac{dv}{v-\frac{mg}{k}} &= -\frac{k}{m}\int dt\\
\ln\left|v-\frac{mg}{k}\right| &= -\frac{k}{m}t+C_0\\
v-\frac{mg}{k} &= Ce^{-\frac{k}{m}t}\\
v &= \frac{mg}{k}+Ce^{-\frac{k}{m}t}
\end{aligned}$$

Solve for C, with $t=0$ and $v=0$.

$$\begin{aligned}
\frac{mg}{k}+Ce^{-\frac{k}{m}\cdot 0} &= 0\\
\frac{mg}{k}+C &= 0\\
C &= -\frac{mg}{k}
\end{aligned}$$

Therefore

$$\begin{aligned}
v(t) &= \frac{mg}{k}-\frac{mg}{k}e^{-\frac{k}{m}t}\\
&= \frac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right).
\end{aligned}$$

(c) Find s by antidifferentiation.

$$\begin{aligned}
s(t) &= \int v(t)dt\\
&= \int\frac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right)dt\\
&= \frac{mg}{k}\left(t+\frac{m}{k}e^{-\frac{k}{m}t}\right)+C^*\\
&= \frac{mg}{k}t+\frac{m^2g}{k^2}e^{-\frac{k}{m}t}+C^*
\end{aligned}$$

Solve for C^*, with $t=0$ and $s=0$.

$$\begin{aligned}
\frac{mg}{k}\cdot 0+\frac{m^2g}{k^2}e^0+C^* &= 0\\
\frac{m^2g}{k^2}+C^* &= 0\\
C^* &= -\frac{m^2g}{k^2}
\end{aligned}$$

Therefore

$$\begin{aligned} s(t) &= \frac{mg}{k}t + \frac{m^2 g}{k^2}e^{-\frac{k}{m}t} - \frac{m^2 g}{k^2} \\ &= \frac{mg}{k}t - \frac{m^2 g}{k^2}\left(1 - e^{-\frac{k}{m}t}\right). \end{aligned}$$

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3. The information given is:

$$\begin{aligned} m\frac{dv}{dt} &= mg - \alpha v^2 \\ v(0) &= 0 \\ \frac{ds}{dt} &= v \\ s(0) &= 0 \end{aligned}$$

(a) At terminal velocity, $\frac{dv}{dt} = 0$, so

$$\begin{aligned} 0 &= mg - av^2 \\ av^2 &= mg \\ v^2 &= \frac{mg}{a} \\ v &= \sqrt{\frac{mg}{a}} \end{aligned}$$

The terminal velocity is $\sqrt{\frac{mg}{a}}$.

(b) Solve for v as a function of t.

$$\begin{aligned} \frac{m}{mg - \alpha v^2}\frac{dv}{dt} &= 1 \\ \frac{dv}{v^2 - \frac{mg}{\alpha}} &= -\frac{\alpha}{m}dt \\ \int \frac{dv}{v^2 - \frac{mg}{\alpha}} &= -\frac{\alpha}{m}\int dt \end{aligned}$$

For convenience, set $\omega := \sqrt{\frac{mg}{\alpha}}$ and substitute.

$$\int \frac{dv}{v^2 - \omega^2} = -\int \frac{g}{\omega^2}dt$$

Decompose $\frac{1}{v^2 - \beta^2}$ into partial fractions.

$$\begin{aligned} \frac{A}{v-\omega} + \frac{B}{v+\omega} &= \frac{1}{v^2 - \omega^2} \\ A(v+\omega) + B(v-\omega) &= 1 \\ (A+B)v + \omega(A-B) &= 0v + 1 \end{aligned}$$

The above yields two linear equations.

$$\begin{array}{rcl} A+B=0 & & \omega(A-B)=1 \\ A=-B & \longrightarrow & \omega(-2B)=1 \\ A=\dfrac{1}{2\omega} & \longleftarrow & B=-\dfrac{1}{2\omega} \end{array}$$

So

$$\begin{aligned} \int \frac{dv}{v^2-\omega^2} &= \frac{1}{2\omega}\left(\int \frac{dv}{v-\omega} - \int \frac{dv}{v+\omega}\right) \\ &= \frac{\ln|v-\omega| - \ln|v+\omega|}{2\omega} + C_0 \end{aligned}$$

The right-hand integral is just $-\frac{g}{\omega^2}t + C_1$, so:

$$\begin{aligned} \frac{\ln|v-\omega| - \ln|v+\omega|}{2\omega} &= -\frac{g}{\omega}t + C_2 \\ \ln\left|\frac{v-\omega}{v+\omega}\right| &= -\frac{2g}{\omega}t + C_3 \\ \frac{v-\omega}{v+\omega} &= Ce^{-\frac{2g}{\omega}t} \end{aligned}$$

Solve for C, with $t=0$ and $v=0$.

$$\begin{aligned} Ce^{-0} &= \frac{0-\omega}{0+\omega} \\ C &= -1 \end{aligned}$$

Continuing:

$$\begin{aligned} \frac{v-\omega}{v+\omega} &= -e^{-\frac{2g}{\omega}t} \\ v-\omega &= -e^{-\frac{2g}{\omega}t}(v+\omega) \\ &= -e^{-\frac{2g}{\omega}t}v - e^{-\frac{2g}{\omega}t}\omega \\ \left(1+e^{-\frac{2g}{\omega}t}\right)v &= \left(1-e^{-\frac{2g}{\omega}t}\right)\omega \\ v &= \omega \cdot \frac{1-e^{-\frac{2g}{\omega}t}}{1+e^{-\frac{2g}{\omega}t}} \end{aligned}$$

Undo the substitution of ω.

$$v(t) = \sqrt{\frac{mg}{\alpha}} \cdot \frac{1-e^{-2\sqrt{\frac{\alpha g}{m}}t}}{1+e^{-2\sqrt{\frac{\alpha g}{m}}t}}$$

(c) Find s by antidifferentiation.

$$\begin{aligned} s &= \int v\,dt \\ &= \omega \int \frac{1-e^{-\frac{2g}{\omega}\cdot t}}{1+e^{-\frac{2g}{\omega}\cdot t}}\,dt \end{aligned}$$

Substitute $u = e^{-\frac{2g}{\omega}t}$, $du = -\frac{2g}{\omega} u\,dt$.

$$\begin{aligned} s &= -\frac{\omega^2}{2g}\int \frac{1-u}{u(1+u)}du \\ &= \frac{\omega^2}{2g}\int \frac{u-1}{u(u+1)}du \end{aligned}$$

Break the integrand into partial fractions.

$$\begin{aligned} \frac{A}{u} + \frac{B}{u+1} &= \frac{u-1}{u(u+1)} \\ A(u+1) + Bu &= u-1 \\ (A+B)u + A &= u-1 \end{aligned}$$

Solve the pair of equations for A and B.

$$\begin{aligned} A+B&=1 \\ -1+B&=1 \quad \leftarrow \quad A=-1 \\ B&=2 \end{aligned}$$

So

$$\begin{aligned} s &= \frac{\omega^2}{2g}\left(-\int \frac{du}{u} + 2\int \frac{du}{u+1}\right) \\ &= \frac{\omega^2}{2g}\left(-\ln(u) + 2\ln(u+1)\right) + C^* \\ &= \frac{\omega^2}{2g}\left(\frac{2g}{\omega}t + 2\ln\left(e^{-\frac{2g}{\omega}t} + 1\right)\right) + C^* \\ &= \omega t + \frac{\omega^2}{g}\ln\left(e^{-\frac{2g}{\omega}t} + 1\right) + C^* \end{aligned}$$

Solve for C^*, with $t = 0$ and $s = 0$.

$$\begin{aligned} 0 + \frac{\omega^2}{g}\ln\left(e^0 + 1\right) + C^* &= 0 \\ \frac{\omega^2}{g} + C^* &= 0 \\ C^* &= -\frac{\omega^2}{g} \end{aligned}$$

Therefore

$$\begin{aligned} s(t) &= \omega t + \frac{\omega^2}{g}\left(\ln\left(e^{-\frac{2g}{\omega}t} + 1\right) - 1\right) \\ &= \omega t - \frac{\omega^2}{g}\left(1 - \ln\left(1 + e^{-\frac{2g}{\omega}t}\right)\right) \end{aligned}$$

Undo the substitution of ω.

$$s(t) = \sqrt{\frac{mg}{\alpha}}\,t - \frac{m}{\alpha}\left(1 - \ln\left(1 + e^{-2\sqrt{\frac{\alpha g}{m}}\,t}\right)\right)$$