Solutions
Section # x
Homework # 1
MATH 527

1.) $\frac{dy}{dt} = 1 + t + y + yt$

$= (1+t) + y(1+t)$

$\frac{dy}{dt} = (1+t)(1+y)$

$\frac{1}{(1+y)} \frac{dy}{dt} = (1+t)$ Now, integrate both sides w.r.t. t, noticing that $\frac{d}{dt}(\ln|1+y|) = \frac{1}{1+y}\frac{dy}{dt}$

$\int \frac{d}{dt}(\ln(1+y))\,dt = \int (1+t)\,dt$

$\ln(1+y) = t + \frac{1}{2}t^2 + c$ Exponentiate both sides.

$1+y = e^{t+\frac{1}{2}t^2+c} = e^{(t+\frac{1}{2}t^2)}e^c = ke^{(t+\frac{1}{2}t^2)}$

$y = ke^{(t+\frac{1}{2}t^2)} - 1.$

2.) $\frac{dy}{dx} = e^{3x+2y}$

$\frac{dy}{dx} = e^{3x}e^{2y}$

$e^{-2y}\frac{dy}{dx} = e^{3x}$ Now, integrate both sides w.r.t. x, noticing that $\frac{d}{dx}(\frac{-1}{2}e^{-2y}) = e^{-2y}\frac{dy}{dt}$

$\int \frac{d}{dx}(-\frac{1}{2}e^{-2y})\,dx = \int e^{3x}\,dx$

$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + c$

$e^{-2y} = \frac{-2}{3}e^{3x} + c_1$ Note: $c_1 = -2c$. Now apply natural log to both sides.

$-2y = \ln(\frac{-2}{3}e^{3x} + c_1)$

$y = \frac{-1}{2}\ln(\frac{-2}{3}e^{3x} + c_1).$

Solutions

3.) $\frac{dy}{dt} = \frac{2t}{y+yt^2}$, $y(2) = 3$

$= \frac{2t}{y(1+t^2)}$

$y\frac{dy}{dt} = \frac{2t}{(1+t^2)}$ Now integrate both sides w.r.t t

$\left(\frac{d}{dt}\left(\frac{1}{2}y^2\right) = y\, dy/dt\right)$

$\int \frac{d}{dt}\left(\frac{1}{2}y^2\right) dt = \int \frac{2t}{(1+t^2)}\, dt$ Let $u = (1+t^2)$, $du = 2t\, dt$

$\frac{1}{2}y^2 = \int \frac{du}{u} = \ln|u| + c = \ln|1+t^2| + c$

$y = \sqrt{2\ln|1+t^2| + c_1}$ Now solve for c_1 using $y(2) = 3$

$3 = \sqrt{2\ln(5) + c_1}$

$9 = 2\ln(5) + c_1 \Rightarrow c_1 = 9 - 2\ln(5)$

So $y = \sqrt{2\ln|1+t^2| + (9 - 2\ln(5))}$

4.) $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$

$(2y-2)\frac{dy}{dx} = 3x^2 + 4x + 2$ Now integrate both sides w.r.t. x

$\left(\frac{d}{dx}(y^2 - 2y) = (2y-2)\, dy/dx\right)$

$\int \frac{d}{dx}(y^2 - 2y)\, dx = \int (3x^2 + 4x + 2)\, dx$

$y^2 - 2y = x^3 + 2x^2 + 2x + c$ Add 1 to both sides to aid factoring.

$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + (c+1)$ Let $c_1 = c+1$

$(y-1)^2 = x^3 + 2x^2 + 2x + c_1$ Now solve for c_1 using $y(0) = -1$

$((-1)-1)^2 = 0^3 + 2(0)^2 + 2(0) + c_1 \Rightarrow 4 = c_1$

So $(y-1)^2 = x^3 + 2x^2 + 2x + 4$

or $y = \left(-\sqrt{x^3+2x^2+2x+4}\right) + 1$. Note: the minus is from $y(0) = -1$.

Solutions

5.) $\cos(y)\frac{dy}{dt} = \frac{-t}{1+t^2}\sin(y)$, $y(1) = \pi/2$

$\frac{\cos(y)}{\sin(y)}\frac{dy}{dt} = \frac{-t}{1+t^2}$

Now integrate w.r.t. t; notice that $\frac{d}{dt}(\ln|\sin(y)|) = \frac{\cos(y)}{\sin(y)}\frac{dy}{dt}$.

$\int \frac{d}{dt}(\ln|\sin(y)|)\,dt = \int \frac{-t}{1+t^2}\,dt$ Let $u = 1+t^2$, $du = 2t\,dt$

$\ln|\sin(y)| = -\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|(1+t^2)| + c$

$\ln|\sin(y)| = -\frac{1}{2}\ln|1+t^2| + c$ Find c by using $y(1) = \pi/2$

$\ln|\sin(\pi/2)| = -\frac{1}{2}\ln|1+(1)^2| + c \Rightarrow 0 = -\frac{1}{2}\ln(2) + c \Rightarrow c = \frac{1}{2}\ln(2)$

So $\ln|\sin(y)| = -\frac{1}{2}\ln|1+t^2| + \frac{1}{2}\ln(2)$ Exponentiate both sides

$\sin(y) = e^{-\frac{1}{2}(\ln|1+t^2| + \ln(2))} = e^{\ln(\sqrt{2(1+t^2)}^{-1})} = (\sqrt{2(1+t^2)}^{-1})$

- $a\ln(x) = \ln(x^a)$
- $\ln(a) + \ln(b) = \ln(ab)$
- $e^{\ln(x)} = x$

So

$y = \sin^{-1}(\sqrt{2(1+t^2)}^{-1})$.

6.) $\frac{dy}{dt} + y\cos(t) = 0$

$dy/dt = -y\cos(t)$

$\frac{1}{y}\frac{dy}{dt} = -\cos(t)$ Now integrate both sides w.r.t. t; notice that $\frac{d}{dt}(\ln|y|) = \frac{1}{y}\frac{dy}{dt}$.

$\int \frac{d}{dt}(\ln|y|)\,dt = -\int \cos(t)\,dt$

$\ln|y| = -\sin(t) + c$

$y = e^{-\sin(t)+c} = e^{-\sin(t)}e^{c} = ke^{-\sin(t)}$.

Solutions

7.) $\frac{dy}{dt} + y = te^t$

Notice that $p(t) = 1$, so $\mu(t) = e^{\int p(t)\,dt} = e^{\int dt} = e^t$

$e^t\left(\frac{dy}{dt} + y\right) = te^t(e^t)$

$e^t\frac{dy}{dt} + e^t y = te^{2t}$ Note that $e^t\frac{dy}{dt} + e^t y = \frac{d}{dt}(e^t y)$

$\frac{d}{dt}(e^t y) = te^{2t}$ Now integrate both sides w.r.t. t.

$\int \frac{d}{dt}(e^t y)\,dt = \int te^{2t}\,dt$ Use int. by parts on RHS:

$u = t$ $\quad v = \frac{1}{2}e^{2t}$

$du = dt$ $\quad dv = e^{2t}\,dt$

$e^t y = \frac{1}{2}te^{2t} - \frac{1}{2}\int e^{2t}\,dt$

$e^t y = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + c$

$y = \frac{1}{2}te^t - \frac{1}{4}e^t + ce^{-t}$

8.) $x\frac{dy}{dx} + 4y = x^3 - x$

$\frac{dy}{dx} + \left(\frac{4}{x}\right)y = x^2 - 1$ Note: $p(x) = \frac{4}{x}$, so $\mu(x) = e^{\int 4/x\,dx} = e^{4\ln(x)}$

$= e^{\ln(x^4)}$

$= x^4$

$x^4\left(\frac{dy}{dx} + \left(\frac{4}{x}\right)y\right) = (x^2-1)(x^4)$

$x^4\frac{dy}{dx} + 4x^3 y = x^6 - x^4$

$\frac{d}{dx}(x^4 y) = x^6 - x^4$ Now integrate both sides w.r.t. x.

$\int \frac{d}{dx}(x^4 y)\,dx = \int (x^6 - x^4)\,dx$

$x^4 y = \frac{1}{7}x^7 - \frac{1}{5}x^5 + C$

$y = \frac{1}{7}x^3 - \frac{1}{5}x + Cx^{-4}$.

Solutions

9.) $x\frac{dy}{dx} + (1+x)y = e^{-x}\sin(2x)$

$\frac{dy}{dx} + (\frac{1}{x}+1)y = (1/x)e^{-x}\sin(2x)$

Note: $p(x) = (1/x + 1)$

so $\mu(x) = e^{\int (1/x+1)\,dx} = e^{(\ln(x)+x)} = e^{\ln(x)}e^{x} = xe^{x}$

$xe^{x}\left(\frac{dy}{dx} + (\frac{1}{x}+1)y\right) = (1/x)e^{-x}\sin(2x)(xe^{x})$

$xe^{x}\frac{dy}{dx} + (e^{x}+xe^{x})y = \sin(2x)$

$\frac{d}{dx}(xe^{x}y) = \sin(2x)$ Now integrate both sides w.r.t. x.

$\int \frac{d}{dx}(xe^{x}y)\,dx = \int \sin(2x)\,dx$

$xe^{x}y = -\frac{1}{2}\cos(2x) + c$

$y = -\frac{1}{2}(\frac{1}{x}e^{-x})\cos(2x) + c(1/x)e^{-x}$

10.) $\cos^2(x)\sin x\frac{dy}{dx} + y\cos^3(x) = 1$

$\frac{dy}{dx} + y\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{1}{\cos^2(x)\sin(x)}$

Note: $p(x) = \left(\frac{\cos(x)}{\sin(x)}\right)$

so $\mu(x) = e^{\int (\frac{\cos(x)}{\sin(x)})dx} = e^{\ln(\sin(x))} = \sin(x)$

$\sin(x)\left(\frac{dy}{dx} + y\frac{\cos(x)}{\sin(x)}\right) = \left(\frac{1}{\cos^2(x)\sin(x)}\right)\sin(x)$

$\sin(x)\frac{dy}{dx} + y\cos(x) = \frac{1}{\cos^2(x)}$

$\frac{d}{dx}(\sin(x)\,y) = \sec^2(x)$ Now integrate both sides w.r.t. x.

$\int \frac{d}{dx}(\sin(x)\,y)\,dx = \int \sec^2(x)\,dx$

$\sin(x)\,y = \tan(x) + c$

$y = \frac{\tan(x)}{\sin(x)} + \frac{c}{\sin(x)} = \frac{1}{\cos(x)} + \frac{c}{\sin(x)} = \sec(x) + c(\csc(x)).$