(1) *Write the system*

$$\begin{aligned} x_1' &= 4x_1 - 3x_2; \\ x_2' &= 2x_1 - 3x_2 \end{aligned}$$

in the form $\mathbf{x}' = A\mathbf{x}$*, and then find the general solution.*

The equation can be written in the given form as

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix} \mathbf{x}.$$

The solutions to this equation will be of form $\mathbf{x} = \mathbf{a}e^{\lambda t}$, where $\mathbf{a}$ is an eigenvector of $A = \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix}$ and λ is the corresponding eigenvalue.

First find the eigenvalues of A.

$$\begin{aligned} |\lambda I - A| &= 0 \\ \begin{vmatrix} \lambda - 4 & 3 \\ -2 & \lambda + 3 \end{vmatrix} &= 0 \\ (\lambda - 4)(\lambda + 3) - (3)(-2) &= 0 \\ \lambda^2 - \lambda - 12 + 6 &= 0 \\ \lambda^2 - \lambda - 6 &= 0 \\ (\lambda - 3)(\lambda + 2) &= 0 \end{aligned}$$

The eigenvalues are $\lambda = 3$ and $\lambda = -2$.

Now find the eigenvectors.

- For $\lambda = 3$:

$$\begin{aligned} \begin{pmatrix} \lambda - 4 & 3 \\ -2 & \lambda + 3 \end{pmatrix} \mathbf{a} &= \mathbf{0} \\ \begin{pmatrix} -1 & 3 \\ -2 & 6 \end{pmatrix} \mathbf{a} &= \mathbf{0} \\ \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \mathbf{a} &= \mathbf{0} \end{aligned}$$

One solution is $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$; the corresponding solution for $\mathbf{x}$ is

$$\mathbf{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} 3e^{3t} \\ e^{3t} \end{pmatrix}.$$

- For $\lambda = -2$:

$$\begin{aligned} \begin{pmatrix} \lambda - 4 & 3 \\ -2 & \lambda + 3 \end{pmatrix} \mathbf{a} &= \mathbf{0} \\ \begin{pmatrix} -6 & 3 \\ -2 & 1 \end{pmatrix} \mathbf{a} &= \mathbf{0} \\ \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} \mathbf{a} &= \mathbf{0} \end{aligned}$$

One solution is $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; the corresponding solution for $\mathbf{x}$ is

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}.$$

The Wronskian for these two particular solutions is

$$\begin{aligned} W &= \begin{vmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{vmatrix} \\ &= \begin{vmatrix} 3e^{3t} & e^{-2t} \\ e^{3t} & 2e^{-2t} \end{vmatrix} \\ &= 6e^t - e^t \\ &= 5e^t \\ &\neq 0. \end{aligned}$$

So the general solution is

$$\mathbf{x} = A \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{3t} + B \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$

That is,

$$\begin{aligned} x_1 &= 3Ae^{3t} + Be^{-2t}; \\ x_2 &= Ae^{3t} + 2Be^{-2t}. \end{aligned}$$

(2) *Find the general solution of the system below:*

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x}.$$

First find the eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

$$\begin{aligned} |\lambda I - A| &= 0 \\ \begin{vmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 3 \end{vmatrix} &= 0 \\ (\lambda - 1)(\lambda - 3) - (-2)(1) &= 0 \\ \lambda^2 - 4\lambda + 3 + 2 &= 0 \\ \lambda^2 - 4\lambda + 5 &= 0 \\ \lambda &= 2 \pm i \end{aligned}$$

The eigenvalues are $\lambda = 2 \pm i$; we'll take just $\lambda = 2 + i$.

Now find the eigenvectors.

$$\begin{aligned} \begin{pmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 3 \end{pmatrix} \mathbf{a} &= \mathbf{0} \\ \begin{pmatrix} 1 + i & -2 \\ 1 & -1 + i \end{pmatrix} \mathbf{a} &= \mathbf{0} \\ \begin{pmatrix} 0 & 0 \\ 1 & -1 + i \end{pmatrix} \mathbf{a} &= \mathbf{0} \; [R_1 := R_1 - (1 \pm i) R_2] \end{aligned}$$

One solution is $\mathbf{a} = \begin{pmatrix} 1 - i \\ 1 \end{pmatrix}$. A particular solution for $\mathbf{x}$ is

$$\begin{aligned} \mathbf{x}_* &= \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} e^{(2+i)t} \\ &= \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} e^{2t} (\cos t + i \sin i) \\ &= \begin{pmatrix} e^{2t} \cos t + ie^{2t} \sin t - ie^{2t} \cos t + e^{2t} \sin t \\ e^{2t} \cos t + ie^{2t} \sin t \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} (\cos t + \sin t) \\ e^{2t} \cos t \end{pmatrix} + i \begin{pmatrix} e^{2t} (\sin t - \cos t) \\ e^{2t} \sin t \end{pmatrix} \end{aligned}$$

So two other particular solutions for $\mathbf{x}$ are

$$\begin{aligned}\mathbf{x}_1 &= \begin{pmatrix} e^{2t}(\cos t + \sin t) \\ e^{2t}\cos t \end{pmatrix}; \\ \mathbf{x}_2 &= \begin{pmatrix} e^{2t}(\sin t - \cos t) \\ e^{2t}\sin t \end{pmatrix}.\end{aligned}$$

The Wronskian for these two particular solutions is

$$\begin{aligned} W &= \begin{vmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{vmatrix} \\ &= \begin{vmatrix} e^{2t}(\cos t + \sin t) & e^{2t}(\sin t - \cos t) \\ e^{2t}\cos t & e^{2t}\sin t \end{vmatrix} \\ &= e^{4t}\sin t(\cos t + \sin t) - e^{4t}\cos t(\sin t - \cos t) \\ &= e^{4t}\left(\sin^2 t - \cos^2 t\right) \\ &\neq 0. \end{aligned}$$

So the general solution is

$$\mathbf{x} = A\begin{pmatrix} e^{2t}(\cos t - \sin t) \\ e^{2t}\cos t \end{pmatrix} + B\begin{pmatrix} e^{2t}(\sin t - \cos t) \\ e^{2t}\sin t \end{pmatrix}.$$

(3) *Find the general solution of the system below:*

$$\mathbf{x}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}\mathbf{x}.$$

First find the eigenvalues of $A = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$.

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(7-\lambda) - (-3)(3) &= 0 \\ \lambda^2 - 8\lambda + 7 + 9 &= 0 \\ \lambda^2 - 8\lambda + 16 &= 0 \\ \lambda &= 4 \end{aligned}$$

The only eigenvalue is $\lambda = 4$.

Now find the eigenvectors.

$$\begin{aligned} \begin{pmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{pmatrix}\mathbf{a} &= \mathbf{0} \\ \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\mathbf{a} &= \mathbf{0} \\ \begin{pmatrix} -3 & -3 \\ 0 & 0 \end{pmatrix}\mathbf{a} &= \mathbf{0}\ [R_2 := R_2 + R_1] \end{aligned}$$

One solution is $\mathbf{a} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$. So one solution for $\mathbf{x}$ is

$$\mathbf{x}_1 = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{4t}.$$

Another solution for $\mathbf{x}$ is of form $\mathbf{x}_2 = \mathbf{a}te^{4t} + \mathbf{b}e^{4t}$, where

$$\begin{aligned}
(A-\lambda I)\,\mathbf{b} &= \mathbf{a} \\
\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\mathbf{b} &= \begin{pmatrix} -3 \\ 3 \end{pmatrix} \\
\begin{pmatrix} -3 & -3 \\ 0 & 0 \end{pmatrix}\mathbf{a} &= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \; [R_2 := R_2 + R_1]
\end{aligned}$$

One option is $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This gives

$$\mathbf{x}_2 = \begin{pmatrix} -3 \\ 3 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}.$$

So the general solution is

$$\mathbf{x} = A\begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{2t} + B\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + B\begin{pmatrix} -3 \\ 3 \end{pmatrix} te^{2t}.$$

(4) *Find the general solution of the system below:*

$$\mathbf{x}' = \begin{pmatrix} 2 & 4 & 4 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}\mathbf{x}.$$

First find the eigenvalues.

$$\begin{aligned}
|\lambda I - A| &= 0 \\
\begin{vmatrix} \lambda-2 & -4 & -4 \\ 1 & \lambda-2 & 0 \\ 1 & 0 & \lambda-2 \end{vmatrix} &= 0 \\
(\lambda-2)^3 + 4(\lambda-2) + 4(\lambda-2) &= 0 \\
(\lambda-2)\left((\lambda-2)^2 + 8\right) &= 0
\end{aligned}$$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 2 + 2\sqrt{2}i$, $\lambda_3 = 2 - 2\sqrt{2}i$. We'll disregard λ_3 since it is just the complex conjugate of λ_2.

Now find the eigenvectors.

- For $\lambda_1 = 2$:

$$\begin{aligned}
\begin{pmatrix} 0 & -4 & -4 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\mathbf{a} &= \mathbf{0} \\
\begin{pmatrix} 0 & -4 & -4 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{a} &= \mathbf{0} \; [R_3 := R_3 - R_2] \\
\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{a} &= \mathbf{0} \; \left[R_1 := \frac{1}{4}R_1\right] \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{a} &= \mathbf{0} \; [R_1 \leftrightarrow R_2]
\end{aligned}$$

A solution is $\mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. The corresponding solution for $\mathbf{x}$ is

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} = \begin{pmatrix} 0 \\ e^{2t} \\ -e^{2t} \end{pmatrix}.$$

- For $\lambda_2 = 2 + 2\sqrt{2}i$:

$$\begin{aligned}
\begin{pmatrix} 2\sqrt{2}i & -4 & -4 \\ 1 & 2\sqrt{2}i & 0 \\ 1 & 0 & 2\sqrt{2}i \end{pmatrix} \mathbf{a} &= \mathbf{0} \\
\begin{pmatrix} 0 & 4 & -4 \\ 1 & 2\sqrt{2}i & 0 \\ 0 & -2\sqrt{2}i & 2\sqrt{2}i \end{pmatrix} \mathbf{a} &= \mathbf{0} \left[R_1 := R_1 - 2\sqrt{2}iR_2;\ R_3 := R_3 - R_2\right] \\
\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2\sqrt{2}i & 0 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{a} &= \mathbf{0} \left[R_1 := \frac{1}{4}R_1;\ R_3 := -\frac{\sqrt{2}}{4}R_3\right] \\
\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2\sqrt{2}i & 0 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{a} &= \mathbf{0} \ [R_1 := R_1 - R_3]
\end{aligned}$$

A solution is $\mathbf{a}_* = \begin{pmatrix} 2\sqrt{2}i \\ -1 \\ -1 \end{pmatrix}$. The corresponding solution for $\mathbf{x}$ is

$$\begin{aligned}
\mathbf{x}_* &= \begin{pmatrix} 2\sqrt{2}i \\ -1 \\ -1 \end{pmatrix} e^{2t} \left(\cos\left(2\sqrt{2}t\right) + i \sin\left(2\sqrt{2}t\right)\right) \\
&= \begin{pmatrix} 2\sqrt{2}ie^{2t}\cos\left(2\sqrt{2}t\right) - 2\sqrt{2}e^{2t}\sin\left(2\sqrt{2}t\right) \\ -e^{2t}\cos\left(2\sqrt{2}t\right) - ie^{2t}\sin\left(2\sqrt{2}t\right) \\ -e^{2t}\cos\left(2\sqrt{2}t\right) - ie^{2t}\sin\left(2\sqrt{2}t\right) \end{pmatrix} \\
&= - \begin{pmatrix} 2\sqrt{2}e^{2t}\sin\left(2\sqrt{2}t\right) \\ e^{2t}\cos\left(2\sqrt{2}t\right) \\ e^{2t}\cos\left(2\sqrt{2}t\right) \end{pmatrix} + i \begin{pmatrix} 2\sqrt{2}e^{2t}\cos\left(2\sqrt{2}t\right) \\ -e^{2t}\sin\left(2\sqrt{2}t\right) \\ -e^{2t}\sin\left(2\sqrt{2}t\right) \end{pmatrix}.
\end{aligned}$$

Thus two other particular solutions are

$$\begin{aligned}
\mathbf{x}_2 &= \begin{pmatrix} 2\sqrt{2}e^{2t}\sin\left(2\sqrt{2}t\right) \\ e^{2t}\cos\left(2\sqrt{2}t\right) \\ e^{2t}\cos\left(2\sqrt{2}t\right) \end{pmatrix}; \\
\mathbf{x}_3 &= \begin{pmatrix} 2\sqrt{2}e^{2t}\cos\left(2\sqrt{2}t\right) \\ -e^{2t}\sin\left(2\sqrt{2}t\right) \\ -e^{2t}\sin\left(2\sqrt{2}t\right) \end{pmatrix}.
\end{aligned}$$

The Wronskian of these three particular solutions is

$$\begin{aligned}
W &= \begin{vmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 2\sqrt{2}e^{2t}\sin\left(2\sqrt{2}t\right) & 2\sqrt{2}e^{2t}\cos\left(2\sqrt{2}t\right) \\ e^{2t} & e^{2t}\cos\left(2\sqrt{2}t\right) & -e^{2t}\sin\left(2\sqrt{2}t\right) \\ -e^{2t} & -e^{2t}\cos\left(2\sqrt{2}t\right) & -e^{2t}\sin\left(2\sqrt{2}t\right) \end{vmatrix} \\
&= \left(2\sqrt{2}e^{6t}\sin^2\left(2\sqrt{2}t\right) - 2\sqrt{2}e^{6t}\cos^2\left(2\sqrt{2}t\right) + 2\sqrt{2}e^{6t}\sin^2\left(2\sqrt{2}t\right) + 2\sqrt{2}e^{6t}\cos^2\left(2\sqrt{2}t\right)\right) \\
&= 4\sqrt{2}e^{6t}\sin^2\left(2\sqrt{2}t\right) \\
&\neq 0.
\end{aligned}$$

So the general solution is

$$\mathbf{x} = \left(A\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + B\begin{pmatrix} 2\sqrt{2}\sin\left(2\sqrt{2}t\right) \\ \cos\left(2\sqrt{2}t\right) \\ \cos\left(2\sqrt{2}t\right) \end{pmatrix} + C\begin{pmatrix} 2\sqrt{2}\cos\left(2\sqrt{2}t\right) \\ -\sin\left(2\sqrt{2}t\right) \\ -\sin\left(2\sqrt{2}t\right) \end{pmatrix}\right)e^{2t}.$$