Exam #2, March 6, 2013
Math 527, University of New Hampshire

Name:
Section: Solutions

INSTRUCTIONS: PLEASE READ CAREFULLY

1. Write your name and section number above. Two points deducted if either is missing or illegible.
2. Show your work and put a box or circle around your answers.
3. Always write equations.
4. Partial credit will be given only if your work is written clearly and in equations.
5. If you have time, check your answers by differentiation and substitution!

Problem 1. (30 pts) Write down the general solutions to these differential equations. Use sines and cosines rather than complex exponentials where applicable. Write down only as much work as you need to solve the problem. k is a constant. Derivatives y', y'' are with respect to x.

(a) $y' + ky = 0$

$y = Ce^{-kx}$

(b) $y'' + k^2 y = 0$

$y(x) = c_1 \cos kx + c_2 \sin kx$

$\lambda^2 + k^2 = 0$

$\lambda^2 = -k^2$

$\lambda = \pm ik$

(c) $y'' - k^2 y = 0$

$\lambda^2 = k^2$

$y(x) = c_1 e^{kx} + c_2 e^{-kx}$

$\lambda = \pm k$

(d) $y'' + 2y' + 5y = 0$

$y(x) = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm \frac{1}{2}\sqrt{-16} = -1 \pm 2i$

(e) $y'' + 6y' + 5y = 0$

$\lambda^2 + 6\lambda + 5 = 0$

$y(x) = c_1 e^{-5x} + c_2 e^{-x}$

$(\lambda + 5)(\lambda + 1) = 0$

(f) $y'' + 6y' + 9y = 0$

$\lambda^2 + 6\lambda + 9 = 0$

$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

$(\lambda + 3)^2 = 0$

Problem 2. (35 pts) Find the general solution of the differential equation. Derivatives y', y'' are with respect to x.

$$y'' + 6y' + 8y = 3e^{-2x} + 2x$$

Homogeneous:

$y'' + 6y' + 8y = 0$

$\lambda^2 + 6\lambda + 8 = 0$

$\lambda = -2, -4$

$y_h = c_1 e^{-2x} + c_2 e^{-4x}$

By Judicious Guessing:

Note that our guess for $3e^{-2x}$ would be $y_{p_1} = Ae^{-2x}$, but this appears in our homogeneous solution, so modify to $y_{p_1} = Axe^{-2x}$

Our guess for $2x$ is $y_{p_2} = Bx + C$.

So our complete guess is $y_p = Axe^{-2x} + Bx + C$

$y_p' = Ae^{-2x} - 2Axe^{-2x} + B$

$y_p'' = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x}$

$= 4Axe^{-2x} - 4Ae^{-2x}$

Substituting into the original ODE yields:

$(4Axe^{-2x} - 4Ae^{-2x}) + 6(Ae^{-2x} - 2Axe^{-2x} + B) + 8(Axe^{-2x} + Bx + C) = 3e^{-2x} + 2x$

$4Axe^{-2x} - 4Ae^{-2x} + 6Ae^{-2x} - 12Axe^{-2x} + 6B + 8Axe^{-2x} + 8Bx + 8C = 3e^{-2x} + 2x$

(e^{-2x} terms): $-4A + 6A = 3 \Rightarrow 2A = 3 \Rightarrow A = 3/2$

(x terms): $8B = 2 \Rightarrow B = 1/4$

(const. terms): $6B + 8C = 0 \Rightarrow 6/4 + 8C = 0 \Rightarrow C = -3/16$

So $y_p = \frac{3}{2}xe^{-2x} + \frac{1}{4}x - \frac{3}{16}$

And our general solution is

$$y(x) = \frac{3}{2}xe^{-2x} + \frac{1}{4}x - \frac{3}{16} + c_1e^{-2x} + c_2e^{-4x}.$$

Note: This problem can also be solved using variation of parameters.

Problem 3. (35 pts) Find the general solution of the differential equation. Derivatives y', y'' are with respect to t.

$$y'' + 4y' + 4y = t^{5/2}e^{-2t}$$

By Variation of Parameters:

$$y_p = u_1 y_1 + u_2 y_2$$

with $u_1 = \int \frac{-y_2 f(t)}{w} dt$, $u_2 = \int \frac{y_1 f(t)}{w} dt$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Homogeneous:

$$y'' + 4y' + 4y = 0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda = -2, -2$$

$$y_1 = e^{-2t} \qquad y_2 = te^{-2t}$$

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{2t} - 2te^{-2t} \end{vmatrix} = e^{-2t}(e^{-2t} - 2te^{-2t}) \cdot te^{-2t}(-2e^{-2t})$$

$$= e^{-4t} - 2te^{-4t} + 2te^{-4t}$$

$$= e^{-4t}$$

$$u_1 = -\int \frac{te^{-2t}(t^{5/2}e^{-2t})}{e^{-4t}} dt = -\int t^{7/2} dt$$

$$= -\frac{2}{9} t^{9/2}$$

$$u_2 = \int \frac{e^{-2t}(t^{5/2}e^{-2t})}{e^{-4t}} dt = \int t^{5/2} dt = \frac{2}{7} t^{7/2}$$

So

$$y_p = e^{-2t}\left(-\frac{2}{9} t^{9/2}\right) + te^{-2t}\left(\frac{2}{7} t^{7/2}\right)$$

$$= e^{-2t}\left(-\frac{2}{9} t^{9/2} + \frac{2}{7} t^{9/2}\right) = t^{9/2}e^{-2t}\left(-\frac{14}{63} + \frac{18}{63}\right)$$

$$= \frac{4}{63} t^{9/2} e^{-2t}$$

And our general solution is:

$$y(t) = \frac{4}{63} t^{9/2} e^{-2t} + c_1 e^{-2t} + c_2 te^{-2t}$$