Corrected Solution for #7

Math 207 ~~Homework #9~~ : Exam #3 Practice 11/12/15

Heaviside + Delta functions Laplace Transforms

1.) $\mathcal{L}\{(3t+1)\,U(t-1)\} = e^{-s}[\mathcal{L}\{3(t+1)+1\} = e^{-s}\mathcal{L}\{3t+4\} = e^{-s}\left[\frac{3}{s^2}+\frac{4}{s}\right]$

2.) $\mathcal{L}\{e^{2t}(t-1)^2\} = \mathcal{L}\{e^{2t}(t^2-2t+1)\} = \frac{2!}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2} = \frac{2-2s+4+s^2-4s+4}{(s-2)^3}$
$= \frac{s^2-6s+10}{(s-2)^3}$

3.) $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+3)-1}{(s+3)^2+25}\right\} = e^{-3t}\left\{\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+25}\right\}\right\} = e^{-3t}\left(2\cos 5t - \frac{1}{5}\sin 5t\right)$

4.) $\mathcal{L}^{-1}\left\{\frac{se^{-\frac{\pi}{2}s}}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+4}\right)\right\} = \cos 2(t-\tfrac{\pi}{2})\,U(t-\tfrac{\pi}{2})$

5.) $f(t) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 0 & 2\pi \le t \end{cases} = \sin t - \sin t\, U(t-2\pi)$
$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin t\, U(t-2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$
$= \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} - e^{-2\pi s}\left(\frac{1}{s^2+1}\right) = \frac{1}{s^2+1}\left(1-e^{-2\pi s}\right)$

6.) $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t^2 & 1 \le t \end{cases} = t^2\,U(t-1)$
$\mathcal{L}\{t^2 U(t-1)\} = e^{-s}\cdot\mathcal{L}\{(t+1)^2\} = e^{-s}\cdot\mathcal{L}\{t^2+2t+1\}$
$= e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$

7.) $y'+2y = f(t);\ y(0)=0,\quad f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t \end{cases}$
$\mathcal{L}\{y'+2y\} = \mathcal{L}\{t - t\,U(t-1)\}$ → was originally incorrectly transformed as $\frac{1}{s}$, not $\frac{1}{s^2}$.
$sF(s) - y(0) + 2F(s) = \frac{1}{s^2} - \mathcal{L}\{t\cdot U(t-1)\} = \frac{1}{s} - e^{-s}\mathcal{L}\{t+1\}$
$F(s)(s+2) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) = \frac{1}{s} - e^{-s}\left(\frac{1+s}{s^2}\right)$
$F(s) = \frac{1}{s^2(s+2)} - e^{-s}\left(\frac{1+s}{s^2(s+2)}\right)$

Partial Fractions

$\frac{1}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2} \Rightarrow 1 = As + 2A + Bs^2 + 2Bs + Cs^2$
$s^2: 0 = B + C \Rightarrow B = -C \Rightarrow \boxed{C = \frac{1}{4}}$
$s^1: 0 = A + 2B \Rightarrow -\frac{1}{2} = 2B \Rightarrow \boxed{B = -\frac{1}{4}}$
$s^0: 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$
$\frac{1+s}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2} \Rightarrow 1+s = As + 2A + Bs^2 + 2Bs + Cs^2$
$s^2: 0 = B + C \Rightarrow B = -C \Rightarrow \boxed{C = -\frac{1}{4}}$
$s^1: 1 = A + 2B \Rightarrow 1 = \frac{1}{2} + 2B \Rightarrow \frac{1}{2} = 2B \Rightarrow \boxed{B = \frac{1}{4}}$
$s^0: 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$
$F(s) = \frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{4}\left(\frac{1}{s+2}\right) - e^{-s}\left[\frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{1}{s+2}\right)\right]$

Inverse Laplace

$y(t) = \frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2t} - U(t-1)\left[\frac{1}{2}(t-1) + \frac{1}{4} - \frac{1}{4}e^{-2(t-1)}\right]$
$y(t) = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4} - U(t-1)\left[\frac{1}{2}t - \frac{1}{4} - \frac{1}{4}e^{-2(t-1)}\right]$
$y(t) = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4} - U(t-1)\left[\frac{1}{4}e^{-2t}\left(e^{2t}(2t-1) - e^{2}\right)\right]$

8.) $y''+2y+y=f(t)$; $y(0)=0$, $y'(0)=1$, $f(t)=\begin{cases}0 & 0\le t<3\\ 2 & 3\le t\end{cases}$

$\mathcal{L}\{y''+2y+y\}=\mathcal{L}\{2\mathcal{U}(t-3)\}$

$s^2F(s)-sy(0)-y'(0)+2sF(s)-2y(0)+F(s)=2\frac{e^{-3s}}{s}$

$F(s)[s^2+2s+1]=2\frac{e^{-3s}}{s}+1$

$F(s)=2e^{-3s}\left(\frac{1}{s(s+1)^2}\right)+\frac{1}{(s+1)^2}$

Partial Fractions $\frac{1}{s(s+1)^2}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2}=\frac{As^2+2As+A+Bs^2+Bs+Cs}{s(s+1)^2}$

$s^2: 0=A+B \Rightarrow A=-B$

$s^1: 0=2A+B+C \Rightarrow 0=A+C$

$s^0: \boxed{1=A} \Rightarrow \boxed{B=-1} \Rightarrow \boxed{C=-1}$

$\frac{1}{s(s+1)^2}=\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}$

Thus, $F(s)=2e^{-3s}\left[\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\right]+\frac{1}{(s+1)^2}$

Inverse Laplace $y(t)=2\mathcal{U}(t-3)\left[1-e^{-(t-3)}-(t-3)e^{-(t-3)}\right]+te^{-t}$

$y(t)=2e^{-t}\mathcal{U}(t-3)\left[e^{t}-e^{3}-te^{3}+3e^{3}\right]+te^{-t}$

$y(t)=2e^{-t}\mathcal{U}(t-3)\left[e^{t}-e^{3}(t-2)\right]+te^{-t}$

9.) $y''+4y'+5y=\delta(t-2\pi)$; $y(0)=y'(0)=0$

Laplace Transform $s^2F(s)+4sF(s)+5F(s)=e^{-2\pi s}$

$F(s)[s^2+4s+5]=e^{-2\pi s}$

$F(s)=e^{-2\pi s}\left(\frac{1}{s^2+4s+5}\right)=e^{-2\pi s}\left(\frac{1}{(s+2)^2+1}\right)$

Inverse Laplace $y(t)=\mathcal{U}(t-2\pi)\left(e^{-2(t-2\pi)}\sin(t-2\pi)\right)=\mathcal{U}(t-2\pi)\left(e^{4\pi-2t}\sin t\right)$