LASZLO KINDRAT
MATH 527, Sec. 04-06.
FALL 2015
HW #8

Partial credit up to the grader (unless specified)

1. $y' - y = 1$, $y(0) = 0$

$s \cdot Y(s) - y(0) - Y(s) = \mathcal{L}\{1\}$

$(s-1)Y(s) = \frac{1}{s}$

$Y(s) = \frac{1}{s(s-1)}$ $\Rightarrow$ Partial fractions:

$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$

$= \frac{A(s-1) + Bs}{s(s-1)} = \frac{(A+B)s - A}{s(s-1)}$

$s^0: 1 = -A \Rightarrow A = -1$

$s^1: 0 = A + B \Rightarrow B = -A = 1$

$Y(s) = -\frac{1}{s} + \frac{1}{s-1}$

$$\boxed{\begin{aligned} y(t) &= -1 + e^t \\ &= e^t - 1 \end{aligned}}$$ $(t \geq 0)$

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2. $y' + 6y = e^{4t}$, $y(0) = 2$

$sY(s) - y(0) + 6Y(s) = \mathcal{L}\{e^{4t}\}$

$(s+6)Y(s) = \frac{1}{s-4} + 2 = \frac{2s-7}{s-4}$

$Y(s) = \frac{2s-7}{(s-4)(s+6)}$ $\Rightarrow$ Partial fractions:

$\frac{2s-7}{(s-4)(s+6)} = \frac{A}{s-4} + \frac{B}{s+6}$

$= \frac{A(s+6) + B(s-4)}{(s-4)(s+6)}$

$= \frac{(A+B)s + (6A-4B)}{(s-4)(s+6)}$

$$s^1: \quad 2 = A + B \quad \Rightarrow \quad A = 2 - B$$
$$s^0: \quad -7 = 6A - 4B$$
$$\hookrightarrow -7 = 6(2-B) - 4B = 12 - 10B$$
$$10B = 12 + 7 \quad \Rightarrow \quad B = 19/10$$
$$Y(s) = \frac{1}{10}\cdot\frac{1}{s-4} + \frac{19}{10}\frac{1}{s+6} \quad \Leftarrow \quad A = 2 - B = 2 - 19/10 = 1/10$$
$$\boxed{y(t) = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}} \qquad (t \geq 0)$$

(3.) $y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$

$$\left[s^2 Y(s) - s\,y(0) - y'(0) + 5s\,Y(s) - 5\,y(0) + 4\,Y(s) = 0\right] \quad +5$$

(Full credit if $y(0)$, $y'(0)$ is evaluated.)

$$(s^2 + 5s + 4)\,Y(s) = s + 5$$
$$Y(s) = \frac{s+5}{s^2+5s+4} = \left[\frac{s+5}{(s+4)(s+1)}\right] \quad +6$$

(+2 for each pole, and the numerator)

Partial fractions:

$$\frac{s+5}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1} = \frac{A(s+1) + B(s+4)}{(s+1)(s+4)} = \frac{(A+B)s + (A+4B)}{(s+1)(s+4)}$$

$$\left.\begin{aligned} 1 &= A + B \\ 5 &= A + 4B \end{aligned}\right\} \ominus \quad 5 - 1 = 4B - B \quad \Rightarrow \quad \left[B = \frac{4}{3}\right] \ +2$$
$$\left[A = 1 - B = -\frac{1}{3}\right] \ +2$$

$$\left[Y(s) = -\frac{1}{3}\frac{1}{s+4} + \frac{4}{3}\frac{1}{s+1}\right] \quad +5$$

Full credit if calculation is correct, w/ incorrect A or B's (or denominators)

+5
$$\boxed{\left[y(t) = -\frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t}\right]} \qquad (t \geq 0)$$

(4.) $y'' + y = \sqrt{2}\sin\sqrt{2}\,t, \quad y(0) = 10, \quad y'(0) = 0$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \sqrt{2}\,\frac{\sqrt{2}}{s^2+2} = \frac{2}{s^2+2}$$

$$(s^2+1)\,Y(s) = \frac{2}{s^2+2} + s\cdot 10$$

$$= \frac{2 + 10s(s^2+2)}{s^2+2}$$

$$Y(s) = \frac{10s^3+20s+2}{(s^2+1)(s^2+2)}$$

Partial fractions:

$$\frac{10s^3+20s+2}{(s^2+1)(s^2+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2}$$

$$= \frac{(As+B)(s^2+2) + (Cs+D)(s^2+1)}{(s^2+1)(s^2+2)}$$

$$= \frac{As^3 + Bs^2 + 2As + 2B + Cs^3 + Ds^2 + Cs + D}{(s^2+1)(s^2+2)}$$

$$= \frac{(A+C)s^3 + (B+D)s^2 + (2A+C)s + (2B+D)}{(s^2+1)(s^2+2)}$$

s^3: $10 = A+C$

s^2: $0 = B+D$

s^1: $20 = 2A+C$

s^0: $2 = 2B+D$

$\ominus$ $20-10 = 2A - A \Rightarrow A = 10 \Rightarrow C = 10 - A = 0$

$\ominus$ $2 - 0 = 2B - B \Rightarrow B = 2 \Rightarrow D = -B = -2$

$$Y(s) = \frac{10s+2}{s^2+1} + \frac{-2}{s^2+2} = 10\cdot\frac{s}{s^2+1} + 2\cdot\frac{1}{s^2+1} - \sqrt{2}\,\frac{\sqrt{2}}{s^2+2}$$

$$\boxed{y(t) = 10\cos t + 2\sin t - \sqrt{2}\sin(\sqrt{2}\,t)} \quad (t \geq 0)$$

⑤ $y'' - 6y' + 9y = t, \quad y(0)=0, \quad y'(0)=1$

$$s^2 Y(s) - s\,y(0) - y'(0) - 6s\,Y(s) + 6y(0) + 9\,Y(s) = \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$(s^2-6s+9)\,Y(s) = \frac{1}{s^2} + 1 = \frac{s^2+1}{s^2}$$

$$Y(s) = \frac{s^2+1}{s^2(s^2-6s+9)} = \frac{s^2+1}{s^2(s-3)^2}$$

Partial fractions:

$$\frac{s^2+1}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$

$$= \frac{As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2}{s^2(s-3)^2}$$

$$= \frac{As^3 - 6As^2 + 9As + Bs^2 - 6Bs + 9B + Cs^3 - 3Cs^2 + Ds^2}{s^2(s-3)^2}$$

$$= \frac{(A+C)s^3 + (-6A+B-3C+D)s^2 + (9A-6B)s + 9B}{s^2(s-3)^2}$$

s^0: $1 = 9B \Rightarrow B = 1/9$

s^1: $0 = 9A - 6B \Rightarrow A = \frac{2}{3}B = \frac{2}{27}$

s^3: $0 = A + C \Rightarrow C = -A = -\frac{2}{27}$

s^2: $1 = -6A + B - 3C + D \Rightarrow D = 1 + 6A - B + 3C = 1 + \frac{4}{9} - \frac{1}{9} - \frac{2}{9} = \frac{10}{9}$

$$Y(s) = \frac{2}{27}\cdot\frac{1}{s} + \frac{1}{9}\cdot\frac{1}{s^2} - \frac{2}{27}\cdot\frac{1}{s-3} + \frac{10}{9}\,\frac{1}{(s-3)^2}$$

$$\boxed{y(t) = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}} \qquad (t \geq 0)$$

⑥ $y'' - 4y' + 4y = t^3 e^{2t}, \quad y(0)=0, \; y'(0)=0$

(+5)

$$\left[s^2 Y(s) - s\,y(0) - y'(0) - 4s\,Y(s) + 4y(0) + 4\,Y(s) = \mathcal{L}\{t^3 e^{2t}\} \right]$$

$$(s^2 - 4s + 4)\,Y(s) = \left[\mathcal{L}\{t^3\}\Big|_{s \to s-2} = \frac{3!}{s^4}\Big|_{s\to s-2} \right] = \left[\frac{3!}{(s-2)^4} \right]$$

(+5) (+5)

$$(s-2)^2\, Y(s) = \frac{3!}{(s-2)^4}$$

$$Y(s) = \frac{3!}{(s-2)^6} = \left[\frac{3!}{s^6}\Big|_{s \to s-2} \right]$$

(+5) (or similar indication of s-translation.)

(+5)

$$\boxed{y(t) = e^{2t}\,\mathcal{L}^{-1}\left\{\frac{3!}{s^6}\right\} = e^{2t}\cdot\frac{1}{4\cdot 5}\,\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\} = \boxed{\frac{1}{20} e^{2t} t^5}} \quad (t \geq 0)$$

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⑦ $y'' - 5y' + 6y = u(t-1), \quad y(0)=0, \quad y'(0)=1$

$$s^2 Y(s) - s\,y(0) - y'(0) - 5s\,Y(s) + 5y(0) + 6\,Y(s) = \frac{e^{-s}}{s}$$

$$(s^2 - 5s + 6)\,Y(s) = \frac{e^{-s}}{s} + 1$$

$$(s-2)(s-3)\,Y(s) = \frac{e^{-s}}{s} + 1$$

$$Y(s) = e^{-s}\cdot\frac{1}{s(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}$$

Partial fractions:

$$\frac{1}{(s-2)(s-3)} = \frac{A}{s-3} + \frac{B}{s-2} = \frac{A(s-2) + B(s-3)}{(s-3)(s-2)} = \frac{(A+B)s - (2A+3B)}{(s-3)(s-2)}$$

s^1: $0 = A + B \Rightarrow B = -A$

s^0: $1 = -(2A + 3B) \Rightarrow 1 = -(2A - 3A) \Rightarrow A = 1 \Rightarrow B = -1$

Partial fractions of the other fraction:

$$\frac{1}{s(s-2)(s-3)} = \frac{C}{s} + \frac{D}{s-2} + \frac{E}{s-3}.$$

$$= \frac{C(s-2)(s-3) + Ds(s-3) + Es(s-2)}{s(s-2)(s-3)}$$

$$= \frac{Cs^2 - 5Cs + 6C + Ds^2 - 3Ds + Es^2 - 2Es}{s(s-2)(s-3)}$$

$$= \frac{(C+D+E)s^2 - (5C+2E+2D)s + 6C}{s(s-2)(s-3)}$$

s^0: $1 = 6C \Rightarrow C = \frac{1}{6}$

s^1: $0 = -(5C + 2E + 3D) \Rightarrow D = -\frac{1}{3}(5C + 2E) = -\frac{5}{18} - \frac{2}{3}E$

s^2: $0 = C + D + E \Rightarrow E = -C - D = -\frac{1}{6} - D$

$$D = -\frac{5}{18} - \frac{2}{3}\left(-\frac{1}{6} - D\right) = -\frac{5}{18} + \frac{2}{18} + \frac{2}{3}D$$

$$\frac{1}{3}D = -\frac{3}{18} = -\frac{1}{6} \Rightarrow D = -\frac{1}{2}$$

$$E = -\frac{1}{6} - D = -\frac{1}{6} + \frac{1}{2} = \frac{1}{3}$$

$$Y(s) = e^{-s}\left(\frac{1}{6}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s-2} + \frac{1}{3}\frac{1}{s-3}\right) + \frac{1}{s-3} - \frac{1}{s-2}$$

$$y(t) = \left[\frac{1}{6} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{3t}\right]_{t \to t-1} u(t-1) + e^{3t} - e^{2t}$$

$$\boxed{= \left(\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right)u(t-1) + e^{3t} - e^{2t}}$$