

Homework #9 : Exam #3 Practice

Heaviside + Delta functions Laplace Transforms

1.) $\mathcal{L}\{(3t+1)u(t-1)\} = e^{-s}[\mathcal{L}\{3(t+1)+1\}] = e^{-s}\mathcal{L}\{3t+4\} = e^{-s}\left[\frac{3}{s^2} + \frac{4}{s}\right]$

2.) $\mathcal{L}\{e^{2t}(t-1)^2\} = \mathcal{L}\{e^{2t}(t^2 - 2t + 1)\} = \frac{2!}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2} = \frac{2-2s+4+s^2-4s+4}{(s-2)^3} = \frac{s^2-6s+10}{(s-2)^3}$

3.) $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+3)-1}{(s+3)^2+25}\right\} = e^{-3t}\left\{\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+25}\right\}\right\} = e^{-3t}\left(2\cos 5t - \frac{1}{5}\sin 5t\right)$

4.) $\mathcal{L}^{-1}\left\{\frac{se^{-\frac{\pi s}{2}}}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi s}{2}}\left(\frac{s}{s^2+4}\right)\right\} = \cos 2\left(t - \frac{\pi}{2}\right)u\left(t - \frac{\pi}{2}\right)$

5.) $f(t) = \begin{cases} \sin t & 0 \leq t < 2\pi \\ 0 & 2\pi \leq t \end{cases} = \sin t - \sin t u(t-2\pi)$
 $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin t u(t-2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} - e^{-2\pi s}\left(\frac{1}{s^2+1}\right) = \frac{1}{s^2+1}(1 - e^{-2\pi s})$

6.) $f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ t^2 & 1 \leq t \end{cases} = t^2 u(t-1)$

6.) $\mathcal{L}\{t^2 u(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\} = e^{-s}\mathcal{L}\{t^2 + 2t + 1\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$

7.) $y' + 2y = f(t); y(0) = 0, f(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & 1 \leq t \end{cases}$

$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{t - t u(t-1)\}$

$sF(s) - y(0) + 2F(s) = \frac{1}{s} - \mathcal{L}\{t \cdot u(t-1)\} = \frac{1}{s} - e^{-s}\mathcal{L}\{t+1\}$

$F(s)(s+2) = \frac{1}{s} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) = \frac{1}{s} - e^{-s}\left(\frac{1+s}{s^2}\right)$

$F(s) = \frac{1}{s(s+2)} - e^{-s}\left(\frac{1+s}{s^2(s+2)}\right)$

Partial Fractions

$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \Rightarrow \frac{1}{s(s+2)} = \frac{As+2A+B}{s(s+2)} \Rightarrow 1 = As+2A+B$

$s^1: 0 = A+B \Rightarrow A = -B$

$s^0: 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}} \Rightarrow \boxed{B = -\frac{1}{2}} \Rightarrow \frac{1}{s(s+2)} = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)$

$\frac{1+s}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2} \Rightarrow \frac{1+s}{s^2(s+2)} = \frac{As+2A+Bs+2Bs+Cs}{s^2(s+2)} \Rightarrow 1+s = As+2A+Bs^2+2Bs+Cs^2$

$s^2: 0 = B+C \Rightarrow B = -C$

$s^1: 1 = A+2B \Rightarrow 1 = \frac{1}{2} + 2B \Rightarrow \frac{1}{2} = 2B \Rightarrow \boxed{B = \frac{1}{4}} \Rightarrow \boxed{C = -\frac{1}{4}}$

$s^0: 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$

$\frac{1+s}{s^2(s+2)} = \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{1}{s+2}\right)$

$F(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+2}\right) - e^{-s}\left(\frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{1}{s+2}\right)\right)$

$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} - u(t-1)\left[\frac{1}{2}(t-1) + \frac{1}{4} - \frac{1}{4}e^{-2(t-1)}\right]$

$y(t) = \frac{1}{2}(1 - e^{-2t}) - u(t-1)\left[\frac{1}{2}t - \frac{1}{4} - \frac{1}{4}e^{-2(t-1)}\right]$

Inverse Laplace

$$8.) y'' + 2y' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1, \quad f(t) = \begin{cases} 0 & 0 \leq t < 3 \\ 2 & 3 \leq t \end{cases}$$

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{2U(t-3)\}$$

$$s^2 F(s) - sy(0) - y'(0) + 2sF(s) - 2y(0) + F(s) = 2 \frac{e^{-3s}}{s}$$

$$F(s)[s^2 + 2s + 1] = 2 \frac{e^{-3s}}{s} + 1$$

$$F(s) = 2e^{-3s} \left(\frac{1}{s(s+1)^2} \right) + \frac{1}{(s+1)^2}$$

Partial Fractions

$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} = \frac{As^2 + 2As + A + Bs^2 + Bs + Cs}{s(s+1)^2}$$

$$s^2: 0 = A + B \Rightarrow A = -B$$

$$s^1: 0 = 2A + B + C \quad \Downarrow \quad 0 = A + C$$

$$s^0: \boxed{1 = A} \Rightarrow \boxed{B = -1} \Rightarrow \boxed{C = -1}$$

$$\frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$\text{Thus, } F(s) = 2e^{-3s} \left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} \right] + \frac{1}{(s+1)^2}$$

Inverse Laplace

$$y(t) = 2U(t-3) \left[1 - e^{-(t-3)} - (t-3)e^{-(t-3)} \right] + te^{-t}$$

$$y(t) = 2e^{-t} U(t-3) \left[e^t - e^3 - te^3 + 3e^3 \right] + te^{-t}$$

$$y(t) = 2e^{-t} U(t-3) \left[e^t - e^3(t-2) \right] + te^{-t}$$

$$9.) y'' + 4y' + 5y = \delta(t - 2\pi); \quad y(0) = y'(0) = 0$$

Laplace Transform

$$s^2 F(s) + 4sF(s) + 5F(s) = e^{-2\pi s}$$

$$F(s)[s^2 + 4s + 5] = e^{-2\pi s}$$

$$F(s) = e^{-2\pi s} \left(\frac{1}{s^2 + 4s + 5} \right) = e^{-2\pi s} \left(\frac{1}{(s+2)^2 + 1} \right)$$

Inverse Laplace

$$y(t) = U(t - 2\pi) \left(e^{-2(t-2\pi)} \sin(t-2\pi) \right) = U(t - 2\pi) \left(e^{4\pi - 2t} \sin t \right)$$