Probs 2 & 4 graded in detail.

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MATH 527
HW #7 Solns
Power Series.

1) Show that $\frac{d}{dx}(\sin(x)) = \cos(x)$

LHS: $\frac{d}{dx}(\sin(x)) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = \left(1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots\right)$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= \cos(x).$$

2) $y'' + k^2 y = 0$

15 – Completed all steps (see ★) of power series method. (even if slightly incorrect)
5 – Correct recurrence relation
2 – Correct final soln
3 – correct comparison w/ $e^{\lambda x}$ soln.

Assume: $y = \sum_{n=0}^{\infty} c_n x^n$

Then: $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

★ Substitute y'' and y into the given DE:

$$y'' + k^2 y = 0 \Rightarrow \left(\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\right) + k^2\left(\sum_{n=0}^{\infty} c_n x^n\right) = 0$$

★ Change the indices of the summation to get matching powers of x: x^m

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{m=n-2} + \underbrace{k^2 \sum_{n=0}^{\infty} c_n x^n}_{m=n} = 0$$

$n = m+2$

$$\sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m + k^2 \sum_{m=0}^{\infty} c_m x^m = 0$$

★ Combine summations now that they have matching powers of x and start at the same value $m=0$.

$$\sum_{m=0}^{\infty} \left[(m+2)(m+1) c_{m+2} + k^2 c_m\right] x^m = 0$$

Because powers of x are linearly independent, the only soln to the eqn

$$\bigstar \quad \sum_{m=0}^{\infty} \left[(m+2)(m+1)c_{m+2} + k^2 c_m\right] x^m = 0 \quad \text{is}$$

that all the coefficients of the power series are zero. Then we have,

$$(m+2)(m+1)c_{m+2} + k^2 c_m = 0 \quad \text{for } m = 0, 1, 2, \ldots$$

$$\bigstar \quad \text{OR} \quad c_{m+2} = \frac{-k^2 c_m}{(m+2)(m+1)} \quad \text{for } m = 0, 1, 2, \ldots$$

$c_0 =$ unknown

$c_1 =$ unknown

$$c_2 = c_{0+2} = \frac{-k^2 c_0}{2 \cdot 1}$$

$$c_3 = c_{1+2} = \frac{-k^2 c_1}{3 \cdot 2}$$

$$c_4 = c_{2+2} = \frac{-k^2 c_2}{4 \cdot 3} = \frac{-k^2}{4 \cdot 3}\left(\frac{-k^2 c_0}{2 \cdot 1}\right) = \frac{k^4 c_0}{4!}$$

$$c_5 = c_{3+2} = \frac{-k^2 c_3}{5 \cdot 4} = \frac{-k^2}{5 \cdot 4}\left(\frac{-k^2 c_1}{3 \cdot 2}\right) = \frac{k^4 c_1}{5!}$$

$$c_6 = c_{4+2} = \frac{-k^2 c_4}{6 \cdot 5} = \frac{-k^2}{6 \cdot 5}\left(\frac{k^4 c_0}{4!}\right) = \frac{-k^6 c_0}{6!}$$

$$c_7 = c_{5+2} = \frac{-k^2 c_5}{7 \cdot 6} = \frac{-k^2}{7 \cdot 6}\left(\frac{k^4 c_1}{5!}\right) = \frac{-k^6 c_1}{7!}$$

Compile terms of the soln. ☆

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 x^0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \dots$$

$$= c_0 \cancel{x^0}^{\,1} + c_1 x + \frac{-k^2 c_0}{2!} x^2 + \frac{-k^2 c_1}{3!} x^3 + \frac{k^4 c_0}{4!} x^4 + \frac{k^4 c_1}{5!} x^5 + \frac{-k^6 c_0}{6!} x^6 + \frac{-k^6 c_1}{7!} + \dots$$

Rearrange into factors of c_0 and c_1

$$= \left[1 - \frac{k^2}{2!} x^2 + \frac{k^4 x^4}{4!} - \frac{k^6 x^6}{6!} + \dots\right] c_0 + \left[x - \frac{k^2}{3!} x^3 + \frac{k^4 x^5}{5!} - \frac{k^6 x^7}{7!} + \dots\right] c_1$$

These power series look like the Maclaurin Series for $\cos(kx)$ & $\sin(kx)$. Need to multiply the second series by $\frac{k}{k}$ to get to proper form.

$$= \left[1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \frac{(kx)^6}{6!} + \dots\right] c_0 + \left[kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \frac{(kx)^7}{7!} + \dots\right] \frac{c_1}{k}$$

$$\boxed{y(x) = \cos(kx)\, c_0 + \sin(kx)\, \hat{c}_1 \quad , \quad \hat{c}_1 = \frac{c_1}{k}}$$

Now compare with soln from ansatz $y = e^{\lambda x}$

$\lambda^2 + k^2 = 0$

$\lambda^2 = -k^2$

$\lambda = \pm ki$

$$y(x) = \tilde{c}_1 e^{+kix} + \tilde{c}_2 e^{-kix}$$

$$\boxed{y(x) = A\cos(kx) + B\sin(kx)}$$

The solutions are identical, with $c_0 = A$ $\hat{c}_1 = \frac{c_1}{k} = B$.

3) $y'' - xy = 0$ ($P(x) = -x$, analytic at every real number.)

Assume:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$ because $x=0$ is an ordinary point

(Thm 6.2.1 guarantees the existence of 2 power series solutions about $x_0 = 0$)

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substitute y and y'' into DE:

$$y'' - xy = 0.$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=0}^{\infty} c_n x^n = 0$$ Update second term in eqn.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{\substack{k=n+1 \\ n=k-1}} = 0$$ Match powers of x.

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$ Rearrange to have the power series start at $k=1$.

$$\underbrace{2 \cdot 1\, c_2 x^0}_{k=0} + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$ Combine series.

$$2 \cdot c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} - c_{k-1} \right] x^k = 0$$

Because powers of x are linearly independent functions the soln to the eqn above is that all coefficients

$\boxed{(k+2)(k+1) c_{k+2} - c_{k-1} = 0. \quad \text{for } k = 1, 2, \ldots}$ & $\boxed{c_2 = 0}$

$$c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)} \quad \text{for } k=1,2,\ldots$$

c_0; c_1 unknown and $c_2 = 0$

recursive relation starts at $k=1 \rightarrow c_3 \sim c_0$

$$c_3 = c_{1+2} \underset{(k=1)}{=} \frac{c_0}{3\cdot 2}$$

$$c_4 = c_{2+2} \underset{(k=2)}{=} \frac{c_1}{4\cdot 3}$$

$$c_5 = c_{3+2} \underset{(k=3)}{=} \frac{c_2}{5\cdot 4} = 0$$

$$c_6 = c_{4+2} \underset{(k=4)}{=} \frac{c_3}{6\cdot 5} = \frac{1}{6\cdot 5}\left(\frac{c_0}{3\cdot 2}\right) = \frac{c_0}{6\cdot 5\cdot 3\cdot 2}$$

$$c_7 = c_{5+2} \underset{(k=5)}{=} \frac{c_4}{7\cdot 6} = \frac{1}{7\cdot 6}\left(\frac{c_1}{4\cdot 3}\right) = \frac{c_1}{7\cdot 6\cdot 4\cdot 3}$$

$$c_8 = c_{6+2} \underset{(k=6)}{=} \frac{c_5}{8\cdot 7} = \frac{1}{8\cdot 7}\left(\frac{c_2}{5\cdot 4}\right) = 0$$

$$c_9 = c_{7+2} \underset{(k=7)}{=} \frac{c_6}{9\cdot 8} = \frac{1}{9\cdot 8}\left(\frac{c_0}{6\cdot 5\cdot 3\cdot 2}\right)$$

$$c_{10} = c_{8+2} \underset{(k=8)}{=} \frac{c_7}{10\cdot 9} = \frac{1}{10\cdot 9}\left(\frac{c_1}{7\cdot 6\cdot 4\cdot 3}\right)$$

$$c_{11} = c_{9+2} \underset{(k=9)}{=} \frac{c_8}{11\cdot 10} = 0$$

Substitute c_n into power series soln.

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

$$= c_0 + c_1 x + 0x^2 + \frac{c_0}{3\cdot 2} x^3 + \frac{c_1}{4\cdot 3} x^4 + 0x^5 + \frac{c_0}{6\cdot 5\cdot 3\cdot 2} x^6 + \frac{c_1}{7\cdot 6\cdot 4\cdot 3} x^7 + 0x^8$$

$$+ \frac{c_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} x^9 + \frac{c_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} x^{10} + 0x^{11} + \dots$$

Rearrange into factors of c_0 and c_1.

$$y(x) = \left(1 + \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} + \frac{x^9}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} + \dots\right) c_0$$

$$+ \left(x + \frac{x^4}{4\cdot 3} + \frac{x^7}{7\cdot 6\cdot 4\cdot 3} + \frac{x^{10}}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} + \dots\right) c_1$$

$$\boxed{y(x) = \left(1 + \sum_{k=1}^{\infty} \frac{1}{2\cdot 3\cdots(3k-1)(3k)} x^{3k}\right) c_0 + \left(x + \sum_{k=1}^{\infty} \frac{1}{3\cdot 4\cdots(3k)(3k+1)} x^{3k+1}\right) c_1}$$

Region of convergence

$y'' - xy = 0$ is already in standard form.

There are no singular solns.

$$\boxed{|x - x_0| = |x - 0| = |x| < \infty}$$

4) $y'' - (x+1)y' - y = 0$ $\quad$ ($P(x) = -(x+1)$, analytic at every real number)

Assume: $y(x) = \sum_{n=0}^{\infty} c_n x^n$ because $x=0$ is an ordinary point.

(Thm 6.2.1 guarantees the existence of two power series solns in this form.)

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

15 - Completed all power series method steps (even if slightly incorrect)
5 - Correct recurrence relation
5 - correct final soln.

Substitute y'', y', y into DE:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - (x+1)\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$ Update second term.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} - \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{k=n} - \underbrace{\sum_{n=1}^{\infty} n c_n x^{n-1}}_{\substack{k=n-1 \\ n=k+1}} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} = 0$$ Match powers of x.

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{\cancel{k=1}\ k=0}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$ Start second series at $k=0$

Now combine series.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) c_{k+2} - k c_k - (k+1) c_{k+1} - c_k \right] x^k = 0.$$

Powers of x are linearly independent, so the soln is

$$(k+2)(k+1) c_{k+2} - k c_k - (k+1) c_{k+1} - c_k = 0 \quad \text{for } k = 0, 1, 2, \ldots$$

$$-(k+1) c_k$$

Three-term recurrence relation

$$c_{k+2} = \frac{(k+1)(c_k + c_{k+1})}{(k+2)(k+1)}$$

$$\boxed{c_{k+2} = \frac{c_k + c_{k+1}}{k+2} \quad \text{for } k = 0, 1, 2.}$$

Express two solns in terms of either just c_0 or just c_1.

$\boxed{c_0 \neq 0, \ c_1 = 0}$

$k=0 \quad c_2 = \dfrac{c_0 + c_1}{2} = \dfrac{c_0}{2}$

$k=1 \quad c_3 = \dfrac{c_1 + c_2}{3} = \dfrac{c_2}{3} = \dfrac{1}{3}\left(\dfrac{c_0}{2}\right)$

$k=2 \quad c_4 = \dfrac{c_2 + c_3}{4} = \dfrac{\frac{c_0}{2} + \frac{c_0}{6}}{4} = \left(\dfrac{2}{3} \cdot \dfrac{1}{4}\right) c_0$

$= \dfrac{1}{6} c_0$

$k=3 \quad c_5 = \dfrac{c_3 + c_4}{5} = \dfrac{\left(\frac{1}{6} + \frac{1}{6}\right) c_0}{5} = \dfrac{1}{15} c_0$

$\boxed{c_0 = 0, \ c_1 \neq 0}$

$k=0 \quad c_2 = \dfrac{c_0 + c_1}{2} = \dfrac{c_1}{2}$

$k=1 \quad c_3 = \dfrac{c_1 + c_2}{3} = \dfrac{c_1 + \frac{c_1}{2}}{3} = \dfrac{3}{2}\left(\dfrac{1}{3}\right) c_1 = \dfrac{1}{2} c_1$

$k=2 \quad c_4 = \dfrac{c_2 + c_3}{4} = \dfrac{\left(\frac{1}{2} + \frac{1}{2}\right) c_1}{4} = \dfrac{1}{4} c_1$

$k=3 \quad c_5 = \dfrac{c_3 + c_4}{5} = \dfrac{\left(\frac{1}{2} + \frac{1}{4}\right) c_1}{5} = \left(\dfrac{3}{4} \cdot \dfrac{1}{5}\right) c_1 = \dfrac{3}{20} c_1$

Compile the terms of the power series soln.

$$y(x) = \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots\right) + \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots\right)$$

$$= \left(c_0 + 0x + \tfrac{1}{2} c_0 x^2 + \tfrac{1}{6} c_0 x^3 + \tfrac{1}{6} c_0 x^4 + \tfrac{1}{15} c_0 x^5 + \dots\right) + \left(c_1 x + \tfrac{1}{2} c_1 x^2 + \tfrac{1}{2} c_1 x^3 + \tfrac{1}{4} c_1 x^4 + \tfrac{3}{20} c_1 x^5 + \dots\right)$$

$$\boxed{y(x) = \left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{15} + \dots\right) c_0 + \left(x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{4} + \frac{3}{20} x^5 + \dots\right) c_1}$$

More on next page.

Specify the region on which the power series solns are guaranteed to converge.

$$y'' - (x+1)y' - y = 0$$ is already in standard form.

No singular points.

Region of convergence: $|x-0| < \infty$

$$\boxed{|x| < \infty}$$

5) IVP $(x-1)y'' - xy' + y = 0$, $y(0) = -2$, $y'(0) = 6$

Standard form: $y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$

$x = 1$ is a singular point.

$x_0 = 0$ is an ordinary point

Thm 6.2.1 guarantees that this DE has two power series solns about $x_0 = 0$. So we assume the following.

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substitute y'', y', y into DE.

$$\underbrace{(x-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}\underbrace{- x\sum_{n=1}^{\infty} n c_n x^{n-1}} + \sum_{n=0}^{\infty} c_n x^n = 0$$ Update first two series.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{\substack{k=n-1\\ n=k+1}} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2\\ n=k+2}} - \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} = 0$$ Match powers of x.

$$\sum_{\cancel{k=1}\ k=0}^{\infty} (k+1)(k)c_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{\cancel{k=1}\ k=0}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

Combine the power series.

$$\sum_{k=0}^{\infty}\left[(k+1)(k)c_{k+1} - (k+2)(k+1)c_{k+2} - kc_k + c_k\right]x^k = 0$$

Because powers of x are linearly independent, the soln is

$$(k+1)(k)c_{k+1} - (k+2)(k+1)c_{k+2} - kc_k + c_k = 0 \quad \text{for } k=0,1,2,\dots$$

$$c_{k+2} = \frac{(+k-1)c_k - k(k+1)c_{k+1}}{-(k+2)(k+1)}$$

$$\boxed{c_{k+2} = \frac{(1-k)c_k + k(k+1)c_{k+1}}{(k+2)(k+1)} \quad \text{for } k=0,1,2,\dots}$$

Three-term recurrence relation. Express two solns in terms of either just c_0 or just c_1.

$\boxed{c_0 \neq 0,\ c_1 = 0}$

$k=0$ $\quad c_2 = \frac{c_0 + 0}{2} = \frac{c_0}{2}$

$k=1$ $\quad c_3 = \frac{0 + 2c_2}{3 \cdot 2} = \frac{1}{3}\left(\frac{1}{2}c_0\right) = \frac{1}{6}c_0$

$k=2$ $\quad c_4 = \frac{-c_2 + 2(3)c_3}{4 \cdot 3} = \frac{-\left(\frac{c_0}{2}\right) + 6\left(\frac{1}{6}c_0\right)}{12} = \frac{\frac{1}{2}c_0}{12} = \frac{1}{24}c_0$

$\boxed{c_0 = 0,\ c_1 \neq 0}$

$k=0$ $\quad c_2 = \frac{c_0 + 0}{2} = 0$

$k=1$ $\quad c_3 = \frac{0 + 2c_2}{3 \cdot 2} = 0$

$k=2$ $\quad c_4 = \frac{-c_2 + 2(3)c_3}{4 \cdot 3} = 0.$

gather terms of the power series solns.

$$y(x) = \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots\right) + \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots\right)$$

$$y(x) = \left(c_0 + 0x + \frac{c_0}{2}x^2 + \frac{c_0}{6}x^3 + \frac{c_0}{24}x^4 + \dots\right) + \left(0 + c_1 x + 0x^2 + 0x^3 + 0x^4 + \dots\right)$$

$$\boxed{y(x) = \left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right)c_0 + xc_1}$$

ordinary point $x_0 = 0$
singular point $x = 1$
Radius of convergence is $R = 1$

1c) $y(0) = -2 \qquad y'(0) = 6$

$\boxed{y(0) = -2}$ $\quad y(0) = 1c_0 + 0c_1 = -2$

$$\boxed{c_0 = -2}$$

$\boxed{y'(0) = 6}$ $\quad y'(x) = \left(\frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \dots\right)c_0 + c_1$

$$y'(0) = 0c_0 + c_1 = 6$$

$$\boxed{c_1 = 6}$$

Final solution

$$y(x) = \left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right)(-2) + 6x$$

which converges at least for $|x| < 1$.

Note: Add and subtract $-2x$.

$$y(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right)(-2) + 8x \Rightarrow y(x) = -2e^x + 8x$$