

$$1) \quad 2y'' - 3y' + y = (t^2 + 1)e^{2t}$$

constant coefficients.  
 2nd order problem  
 non-homog.

ansatz  $y_c = e^{\lambda t}$

$$2\lambda^2 - 3\lambda + 1 = 0$$

$$(2\lambda - 1)(\lambda - 1) = 0$$

$$\lambda = \frac{1}{2}, 1 \quad f(t) = \frac{(t^2 + 1)e^{2t}}{2}$$

$$y_c(t) = c_1 e^{\frac{1}{2}t} + c_2 e^t$$

$$y_p(t) = u_1(t)e^{\frac{1}{2}t} + u_2(t)e^t$$

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{\frac{1}{2}t} & e^t \\ \frac{1}{2}e^{\frac{1}{2}t} & e^t \end{vmatrix}$$

$$W = e^{\frac{1}{2}t+t} - \frac{1}{2}e^{\frac{1}{2}t+t}$$

$$W = \frac{1}{2}e^{\frac{1}{2}t+t} = \frac{1}{2}e^{\frac{3}{2}t}$$

$$u_1'(t) = \frac{-y_2(t)f(t)}{W}$$

$$u_1'(t) = \frac{-e^t(t^2+1)e^{2t}(\frac{1}{2})}{\frac{1}{2}e^{\frac{3}{2}t}}$$

$$u_1'(t) = \frac{-2(t^2+1)e^{\frac{3}{2}t}}{2}$$

$$u_1(t) = - \int (t^2+1)e^{\frac{3}{2}t} dt$$

IBP  $u = t^2 + 1 \quad v = \frac{2}{3}e^{\frac{3}{2}t}$

$$du = 2t dt \quad dv = e^{\frac{3}{2}t} dt$$

$$u_1(t) = - \left( (t^2+1)\frac{2}{3}e^{\frac{3}{2}t} - \int \frac{2}{3}e^{\frac{3}{2}t} 2t dt \right)$$

$$- \frac{4}{3} \int te^{\frac{3}{2}t} dt$$

IBP  $u = t \quad v = \frac{2}{3}e^{\frac{3}{2}t}$

$$du = dt \quad dv = e^{\frac{3}{2}t} dt$$

$$- \left[ \frac{2}{3}(t^2+1)e^{\frac{3}{2}t} - \frac{4}{3} \left( t\frac{2}{3}e^{\frac{3}{2}t} - \int \frac{2}{3}e^{\frac{3}{2}t} dt \right) - \frac{4}{9}e^{\frac{3}{2}t} \right]$$

$$u_1(t) = -\frac{2}{3}(t^2+1)e^{\frac{3}{2}t} + \frac{8}{9}te^{\frac{3}{2}t} - \frac{16}{27}e^{\frac{3}{2}t}$$

$$u_2'(t) = \frac{y_1(t)f(t)}{W} = \frac{e^{\frac{1}{2}t}(t^2+1)e^{2t}(\frac{1}{2})}{\frac{1}{2}e^{\frac{3}{2}t}}$$

$$u_2'(t) = (t^2+1)e^t$$

$$u_2(t) = \int (t^2+1)e^t dt$$

IBP  $u = t^2 + 1 \quad v = e^t$

$$du = 2t dt \quad dv = e^t dt$$

$$u_2(t) = (t^2+1)e^t - \int e^t 2t dt$$

$$u_2(t) = \left[ (t^2+1)e^t - 2 \int te^t dt \right]$$

IBP  $u = t \quad v = e^t$   
 $du = dt \quad dv = e^t dt$

$$- \left( te^t - \int e^t dt \right) - e^t$$

$$u_2(t) = \left[ (t^2+1)e^t - 2te^t + 2e^t \right]$$

$$y_p(t) = \left( -\frac{2}{3}(t^2+1)e^{\frac{3}{2}t} + \frac{8}{9}te^{\frac{3}{2}t} - \frac{16}{27}e^{\frac{3}{2}t} \right) e^{\frac{1}{2}t}$$

$$+ \left( -(t^2+1)e^t - 2te^t + 2e^t \right) e^t$$

$$y_p(t) = \left( -\frac{2}{3}(t^2+1) + \frac{8}{9}t - \frac{16}{27} \right) e^{2t}$$

$$+ \left( -(t^2+1) - 2t + 2 \right) e^{2t}$$

$$y_p(t) = \left[ \left( -\frac{2}{3} + \frac{2}{3} \right) (t^2+1) + \left( \frac{8}{9} - \frac{18}{9} \right) t \right.$$

$$\left. + \left( -\frac{2}{3} - \frac{16}{27} + 1 + 2 \right) \right] e^{2t}$$

$$y_p(t) = \left[ \frac{1}{3}(t^2+1) - \frac{10}{9}t \right.$$

$$\left. + \left( -\frac{18}{27} - \frac{16}{27} + \frac{27}{27} + \frac{54}{27} \right) \right] e^{2t}$$

$$y_p(t) = \left[ \frac{1}{3}(t^2+1) - \frac{10}{9}t + \frac{47}{27} \right] e^{2t}$$

$$y_p(t) = \left( \frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27} \right) e^{2t}$$

$$y_{gen}(t) = c_1 e^{\frac{1}{2}t} + c_2 e^t$$

$$+ \left( \frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27} \right) e^{2t}$$

$$2) y'' + y = \sec(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$y_c(t) = e^{\lambda t}$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$y_c(t) = c_1 e^{it} + c_2 e^{-it} \text{ OR}$$

$$y_c(t) = \tilde{c}_1 \cos(t) + \tilde{c}_2 \sin(t)$$

$$y_p(t) = u_1(t) \cos(t) + u_2(t) \sin(t)$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$W = \cos^2 t + \sin^2 t = 1$$

$$u_1'(t) = \frac{-y_2(t) f(t)}{W} \\ = \frac{-\sin(t) \sec(t)}{1}$$

$$u_1(t) = - \int \frac{\sin(t)}{\cos(t)} dt$$

$$= \begin{aligned} u &= \cos(t) \\ du &= -\sin(t) dt \end{aligned}$$

$$u_1(t) = \int \frac{1}{u} dt = \ln|\cos(t)| + \overset{0}{C}$$

$$u_1(t) = \ln|\cos(t)| \quad \cos(t) > 0 \text{ for}$$

$$\boxed{u_1(t) = \ln(\cos(t))} \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$u_2'(t) = \frac{y_1(t) f(t)}{W} = \frac{\cos(t) \sec(t)}{1}$$

$$u_2'(t) = 1$$

$$u_2(t) = \int 1 dt$$

$$u_2(t) = t + \overset{0}{C}$$

$$\boxed{u_2(t) = t}$$

$$\boxed{y_p(t) = \ln(\cos(t)) \cos(t) + t \sin(t)}$$

$$\boxed{y_{\text{gen}}(t) = \tilde{c}_1 \cos(t) + \tilde{c}_2 \sin(t) \\ + \ln(\cos(t)) \cos(t) + t \sin(t)}$$

$$3) y'' - 3y' + 2y = te^{3t} + 1$$

$$y_c(t) = e^{\lambda t}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, 2$$

$$y_c(t) = c_1 e^t + c_2 e^{2t}$$

$$y_p(t) = u_1 e^t + u_2 e^{2t}$$

$$W = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = 2e^{3t} - e^{3t}$$

$$W = e^{3t}$$

$$u_1'(t) = \frac{-y_2(t) f(t)}{W} = \frac{-e^{2t}(te^{3t} + 1)}{e^{3t}}$$

$$= -(te^{3t} + 1)e^{-t}$$

$$u_1'(t) = -te^{2t} - e^{-t}$$

$$u_1(t) = -\int te^{2t} dt - \int e^{-t} dt$$

IBP  $u = t \quad v = \frac{1}{2}e^{2t}$   
 $du = dt \quad dv = e^{2t} dt$

$$-\left(\frac{1}{2}te^{2t} - \int \frac{1}{2}e^{2t} dt\right) - \int e^{-t} dt$$

$$u_1 = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t} + \varphi^0$$

$$u_1(t) = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t}$$

$$u_2'(t) = \frac{y_1(t)f(t)}{W} = e^t(te^{3t} + 1)e^{-3t} = (te^{4t} + e^t)e^{-3t} = te^t + e^{-2t}$$

$$u_2(t) = \int te^t dt + \int e^{-2t} dt$$

$$u_2(t) = te^t - e^t + -\frac{1}{2}e^{-2t} + \varphi^0$$

$$u_2(t) = te^t - e^t - \frac{1}{2}e^{-2t}$$

$$y_p(t) = \left(-\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t}\right)e^t + \left(te^t - e^t - \frac{1}{2}e^{-2t}\right)e^{2t}$$

$$y_p(t) = -\frac{1}{2}te^{3t} + \frac{1}{4}e^{3t} + e^0$$

$$+ te^{3t} - e^{3t} - \frac{1}{2}e^0 \rightarrow \cancel{0}$$

$$y_p(t) = \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} + e^{2t} + \frac{1}{2}$$

$$y_{gen'l}(t) = c_1 e^t + c_2 e^{2t} + \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} + \frac{1}{2}$$

$$4) 3y'' + 4y' + y = e^{-t} \sin t, \quad y(0) = 1$$

$$y'(0) = 0$$

$$y_c(t) = e^{\lambda t}$$

$$3\lambda^2 + 4\lambda + 1 = 0$$

$$(3\lambda + 1)(\lambda + 1) = 0$$

$$\lambda = -\frac{1}{3}, -1$$

$$y_c(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$y_p(t) = u_1(t) e^{-\frac{1}{3}t} + u_2(t) e^{-t}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{1}{3}t} & e^{-t} \\ -\frac{1}{3}e^{-\frac{1}{3}t} & -e^{-t} \end{vmatrix}$$

$$W = -e^{-\frac{1}{3}t-t} + \frac{1}{3}e^{-\frac{1}{3}t-t}$$

$$W = -\frac{2}{3}e^{-\frac{4}{3}t}$$

$$u_1'(t) = \frac{-y_2(t)f(t)}{W} = \frac{-e^{-t} e^{-t} \sin t}{-\frac{2}{3}e^{-\frac{4}{3}t}}$$

$$u_1'(t) = \frac{1}{3} \frac{3}{2} \sin t e^{-2t + \frac{4}{3}t}$$

$$u_1'(t) = \frac{1}{2} \sin t e^{-\frac{2}{3}t}$$

$$u_1(t) = \frac{1}{2} \int \sin t e^{-\frac{2}{3}t} dt$$

$$u = \sin t \quad v = -\frac{3}{2} e^{-\frac{2}{3}t}$$

$$du = \cos t dt \quad dv = e^{-\frac{2}{3}t} dt$$

$$= \frac{1}{2} \left( -\frac{3}{2} \sin t e^{-\frac{2}{3}t} - \int -\frac{3}{2} e^{-\frac{2}{3}t} \cos t dt \right)$$

$$u = \cos t \quad v = -\frac{3}{2} e^{-\frac{2}{3}t}$$

$$du = -\sin t dt \quad dv = e^{-\frac{2}{3}t} dt$$

$$u_1(t) = \frac{1}{2} \left( -\frac{3}{2} \sin t e^{-\frac{2}{3}t} + \frac{3}{2} \cos t \left( \frac{3}{2} e^{-\frac{2}{3}t} \right) - \int +\frac{3}{2} e^{-\frac{2}{3}t} \sin t dt \right)$$

$$\text{Let } I = \int \sin t e^{-\frac{2}{3}t} dt$$

$$u_1(t) = \frac{1}{2} I$$

$$\frac{1}{2} I = \frac{1}{2} \left( -\frac{3}{2} \sin t e^{-\frac{2}{3}t} - \frac{9}{4} \cos t e^{-\frac{2}{3}t} - \frac{9}{4} I \right)$$

$$\left( \frac{4}{4} + \frac{9}{4} \right) I = \frac{3}{2} \left( -\frac{3}{2} \cos t - \sin t \right) e^{-\frac{2}{3}t}$$

$$I = \frac{4}{13} \cdot \frac{3}{2} \left( \frac{3}{2} \cos t - \sin t \right) e^{-\frac{2}{3}t}$$

$$I = \frac{6}{13} \left( \frac{3}{2} \cos t - \sin t \right) e^{-\frac{2}{3}t} + C^0$$

$$u_1(t) = \frac{1}{2} I = \frac{3}{13} \left( \frac{3}{2} \cos t - \sin t \right) e^{-\frac{2}{3}t}$$

$$u_2'(t) = \frac{y_1(t)f(t)}{W} = \frac{e^{-\frac{1}{3}t} \left( e^{-t} \sin t \frac{1}{3} \right)}{-\frac{2}{3} e^{-\frac{4}{3}t}}$$

$$= -\frac{1}{3} \frac{3}{2} \sin t e^{-\frac{1}{3}t-t+\frac{4}{3}t}$$

$$u_2'(t) = -\frac{1}{2} \sin t e^{0t}$$

$$u_2(t) = -\frac{1}{2} \int \sin t dt$$

$$u_2(t) = -\frac{1}{2} (-\cos t + C^0)$$

$$u_2(t) = \frac{1}{2} \cos t$$

$$y_{\text{gen'l}}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$+ \left( \frac{3}{13} \left( \frac{2}{2} \cos(t) - \sin(t) \right) e^{-\frac{2}{3}t} \right) e^{-\frac{1}{3}t}$$

$$+ \left( \frac{1}{2} \cos(t) \right) e^{-t}$$

$$y_{\text{gen'l}}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$+ \frac{3}{13} \left( \frac{2}{2} \cos(t) - \sin(t) \right) e^{-t}$$

$$+ \left( \frac{1}{2} \cos(t) \right) e^{-t}$$

$$y_{\text{gen'l}}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$+ \left[ \left( \frac{9}{26} + \frac{13}{26} \right) \cos(t) - \frac{3}{13} \sin(t) \right] e^{-t}$$

$$y_{\text{gen'l}}(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-t}$$

$$+ \left( \frac{2}{13} \cos(t) - \frac{3}{13} \sin(t) \right) e^{-t}$$

IC)  $y(0) = 1, y'(0) = 0$

$y(0) = 1$

$$1 = c_1 + c_2 + \frac{2}{13}$$

$$c_1 = \frac{11}{13} - c_2$$

$y'(0) = 0$

$$y'(t) = c_1 \left( -\frac{1}{3} \right) e^{-\frac{1}{3}t} - c_2 e^{-t}$$

$$+ \left( \frac{2}{13} \cos(t) - \frac{3}{13} \sin(t) \right) e^{-t} + e^{-t} \left( -\frac{2}{13} \sin(t) - \frac{3}{13} \cos(t) \right)$$

$y'(0) = 0$

$$0 = -\frac{1}{3} c_1 - c_2 - \frac{2}{13} - \frac{3}{13}$$

$$0 = -\frac{1}{3} \left( \frac{11}{13} - c_2 \right) - c_2 - \frac{5}{13}$$

$$\frac{5}{13} + \frac{11}{39} = \left( +\frac{1}{3} - 1 \right) c_2$$

$$\frac{15}{39} + \frac{11}{39}$$

$$\frac{26}{39}$$

$$\frac{2}{3} = -\frac{2}{3} c_2$$

$$c_2 = -1 \rightarrow c_1 = \frac{11}{13} - \left( -\frac{13}{13} \right)$$

$$c_1 = \frac{24}{13}$$

$$y(t) = \frac{24}{13} e^{-\frac{1}{3}t} - e^{-t} + \left( \frac{2}{13} \cos(t) - \frac{3}{13} \sin(t) \right) e^{-t}$$

$$y(t) = \frac{1}{13} e^{-t} \left( 24 e^{\frac{2}{3}t} - 13 + 2 \cos(t) - 3 \sin(t) \right)$$