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MATH 527, Sec. 04-06
HW #5

Partial credit within each group is up to the grader.

(1.) $6y'' - 7y' + y = 0$

$6\lambda^2 - 7\lambda + 1 = 0$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{(-7)^2 - 4\cdot 6\cdot 1}}{2\cdot 6} = \frac{7 \pm 5}{12} \Rightarrow \lambda_1 = \frac{7+5}{12} = 1, \quad \lambda_2 = \frac{7-5}{12} = \frac{1}{6}$$

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{t} + c_2 e^{t/6}}$$

(2.) $y'' + 2y' + 3y = 0$

$\lambda^2 + 2\lambda + 3 = 0$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 3}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm i\sqrt{2} \Rightarrow \lambda_1 = -1 + i\sqrt{2}, \quad \lambda_2 = -1 - i\sqrt{2}$$

$\lambda_{1,2}$ are complex: let $\alpha = -1$, $\beta = \sqrt{2}$

$\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

$$\boxed{y(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t) = e^{-t}(A\cos\sqrt{2}t + B\sin\sqrt{2}t)}$$

or $y(t) = c_1 e^{-1+i\sqrt{2}} + c_2 e^{-1-i\sqrt{2}}$

(3.) $y'' - 6y' + 9y = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{(-6)^2 - 4\cdot 1\cdot 9}}{2} = \frac{6 \pm 0}{2} = 3 \Rightarrow \lambda_1 = \lambda_2 = 3$$

$\lambda = \lambda_{1,2} = 3$ Double root, thus:

$$\boxed{y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} = c_1 e^{3t} + c_2 t e^{3t}}$$

4. $2y'' + y' - 10y = 0 \qquad y(1) = 5,\ y'(1) = 2$

$2\lambda^2 + \lambda - 10 = 0$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot (-10)}}{2 \cdot 2} = \frac{-1 \pm \sqrt{81}}{4} = \frac{-1 \pm 9}{4} \Rightarrow \begin{array}{l} \lambda_1 = \frac{-1+9}{4} = 2 \\ \lambda_2 = \frac{-1-9}{4} = -\frac{5}{2} \end{array}$$

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{2t} + c_2 e^{-\frac{5t}{2}}$$

Applying ICs:

$$y(1) = c_1 e^2 + c_2 e^{-5/2} = 5 \quad \Rightarrow c_2 e^{-5/2} = 5 - c_1 e^2$$

$$y'(1) = \left[c_1 \cdot 2 \cdot e^{2t} + c_2 \left(-\frac{5}{2}\right) e^{-\frac{5t}{2}}\right]_{t=1} = 2e^2 c_1 - \frac{5}{2} c_2 e^{-5/2} = 2$$

Solve for c_1, c_2:

$$2e^2 c_1 - \frac{5}{2}\left(5 - c_1 e^2\right) = 2$$

$$2e^2 c_1 - \frac{25}{2} + \frac{5}{2} e^2 c_1 = 2 \qquad / + \frac{25}{2}$$

$$\frac{9}{2} e^2 c_1 = \frac{29}{2}$$

$$e^2 c_1 = \frac{29}{9} \quad \Rightarrow c_1 = \frac{29}{9} e^{-2}$$

$$c_2 e^{-5/2} = 5 - \frac{29}{9} = -\frac{16}{9} \quad \Rightarrow c_2 = -\frac{16}{9} e^{5/2}$$

Particular solution:

$$\boxed{y(t) = \frac{29}{9} e^{-2} e^{2t} - \frac{16}{9} e^{5/2} e^{-\frac{5t}{2}} = \frac{29}{9} e^{2(t+1)} - \frac{16}{9} e^{\frac{5}{2}(1-t)}}$$

(5.) $4y'' - 4y' + y = 0 \qquad y(0) = 0, \; y'(0) = 3$

$4\lambda^2 - 4\lambda + 1 = 0$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4\cdot 4\cdot 1}}{2\cdot 4} = \frac{4 \pm 0}{2\cdot 4} = \frac{1}{2} \Rightarrow \left[\lambda_1 = \lambda_2 = \frac{1}{2}\right] \; +5$$

$\lambda = \lambda_{1,2} = \frac{1}{2}$ Double root, thus:

$$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} = \left[c_1 e^{t/2} + c_2 t e^{t/2}\right] \; +5$$

Applying ICs:

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 \cdot 1 = [c_1 = 0] \; +5$$

so $y(t) = c_2 t e^{t/2}$

$$y'(0) = \left[c_2 e^{\lambda t} + c_2 t \cdot \frac{1}{2} e^{t/2}\right]_{t=0} = c_2 \cdot 1 + c_2 \cdot 0 \cdot \frac{1}{2} \cdot 1 = [c_2 = 3] \; +5$$

Particular solution:

$$\boxed{y(t) = 3te^{t/2}} \; +5$$

⑥ $y'' + y' + 2y = 0, \quad y(0) = 1, \ y'(0) = -2$

$\lambda^2 + \lambda + 2 = 0$

$\lambda_{1,2} = \dfrac{-1 \pm \sqrt{1^2 - 4\cdot 1 \cdot 2}}{2} = \dfrac{-1 \pm \sqrt{-7}}{2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{7}}{2} \Rightarrow \begin{matrix}\lambda_1 = -\frac{1}{2} + i\frac{\sqrt{7}}{2} \\ \lambda_2 = -\frac{1}{2} - i\frac{\sqrt{7}}{2}\end{matrix}$

Solution 1: let $\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{7}}{2}$, so $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

$y(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t) = e^{-t/2}\left(A\cos\dfrac{\sqrt{7}t}{2} + B\sin\dfrac{\sqrt{7}t}{2}\right)$

Applying ICs:

$y(0) = 1(A\cdot 1 + B\cdot 0) = A = 1 \Rightarrow y(t) = e^{-t/2}\left(\cos\dfrac{\sqrt{7}t}{2} + B\sin\dfrac{\sqrt{7}t}{2}\right)$

$y'(0) = \left[-\dfrac{1}{2}e^{-t/2}\left(\cos\dfrac{\sqrt{7}t}{2} + B\sin\dfrac{\sqrt{7}t}{2}\right) + e^{-t/2}\left(-\dfrac{\sqrt{7}}{2}\sin\dfrac{\sqrt{7}t}{2} + \dfrac{\sqrt{7}}{2}B\cos\dfrac{\sqrt{7}t}{2}\right)\right]_{t=0}$

$= -\dfrac{1}{2}\cdot 1\cdot(1 + B\cdot 0) + 1\cdot\left(-\dfrac{\sqrt{7}}{2}\cdot 0 + \dfrac{\sqrt{7}}{2}\cdot B\cdot 1\right)$

$= -\dfrac{1}{2} + \dfrac{\sqrt{7}}{2}B = -2 \quad \Rightarrow B = -\dfrac{3}{\sqrt{7}} = -\dfrac{3}{7}\sqrt{7}$

$\boxed{y(t) = e^{-t/2}\left(\cos\dfrac{\sqrt{7}t}{2} - \dfrac{3}{7}\sqrt{7}\sin\dfrac{\sqrt{7}t}{2}\right)}$

Solution 2: $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{\left(-\frac{1}{2} + i\frac{\sqrt{7}}{2}\right)t} + c_2 e^{\left(\frac{1}{2} - i\frac{\sqrt{7}}{2}\right)t}$

Applying ICs:

$y(0) = c_1\cdot 1 + c_2\cdot 1 = c_1 + c_2 = 1$

$y'(0) = \left[c_1\left(-\frac{1}{2} + i\frac{\sqrt{7}}{2}\right)e^{\left(-\frac{1}{2} + i\frac{\sqrt{7}}{2}\right)t} + c_2\left(-\frac{1}{2} - i\frac{\sqrt{7}}{2}\right)e^{\left(\frac{1}{2} - i\frac{\sqrt{7}}{2}\right)t}\right]_{t=0}$

$= c_1\left(-\frac{1}{2} + i\frac{\sqrt{7}}{2}\right) + c_2\left(-\frac{1}{2} - i\frac{\sqrt{7}}{2}\right) = -\frac{1}{2}\underbrace{(c_1 + c_2)}_{1} + \frac{\sqrt{7}}{2}i(c_1 - c_2) = -2$

$c_1 - c_2 = \frac{3}{\sqrt{7}}i$

$c_1 = \frac{1}{2} + \frac{3}{2\sqrt{7}}i, \ c_2 = \frac{1}{2} - \frac{3}{2\sqrt{7}}i \Rightarrow \boxed{y(t) = \left(\frac{1}{2} + \frac{3i}{2\sqrt{7}}\right)e^{\left(-\frac{1}{2} + i\frac{\sqrt{7}}{2}\right)t} + \left(\frac{1}{2} - \frac{3i}{2\sqrt{7}}\right)e^{\left(-\frac{1}{2} - i\frac{\sqrt{7}}{2}\right)t}}$

(7.) $y'' + 9y = 0 \qquad y(0) = 2, \quad y'(0) = -3/2$

$\lambda^2 + 9 = 0 \quad \Rightarrow \lambda_{1,2} = \pm\sqrt{-9} = \pm 3i \quad \Rightarrow \boxed{\lambda_1 = 3i,\ \lambda_2 = -3i}$ +5

let $\alpha = 0,\ \beta = 3 \quad \Rightarrow \lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$

a) $\boxed{y(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t) = A\cos 3t + B\sin 3t}$ +5

b) $y(0) = A\cdot 1 + B\cdot 0 = A = 2$

$y'(0) = \left[-3A\sin 3t + 3B\cos 3t\right]_{t=0} = -3\cdot 0 + 3\cdot B\cdot 1 = 3B = -3/2$

Particular solution: $\qquad B = -\frac{1}{2}$

$\boxed{y(t) = 2\cos 3t - \frac{1}{2}\sin 3t}$ +5

c) $\boxed{y(t) = (c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}) = c_1 e^{3it} + c_2 e^{-3it}}$ +5

d) $y(0) = c_1\cdot 1 + c_2\cdot 1 = c_1 + c_2 = 2$

$y'(0) = \left[c_1 3i\, e^{3it} + c_2(-3i)e^{-3it}\right]_{t=0} = 3ic_1 - 3ic_2 = -3/2$

$c_1 - c_2 = \frac{i}{2}$

$2c_1 = 2 + \frac{i}{2} \Rightarrow c_1 = 1 + \frac{i}{4}, \quad c_2 = 2 - c_1 = 1 - \frac{i}{4}$

$y(t) = \left(1 + \frac{i}{4}\right)e^{3it} + \left(1 - \frac{i}{4}\right)e^{-3it}$

$= 1(e^{3it} + e^{-3it}) + \frac{i}{4}(e^{3it} - e^{-3it})$ / Use Euler's formula

$= 1\cdot(2\cos 3t) + \frac{i}{4}(2i\sin 3t)$

$\boxed{y(t) = 2\cos 3t - \frac{1}{2}\sin 3t}$ +5

(8.) $(\cos x + i\sin x)^n \underset{\substack{\uparrow \\ \text{E's form.}}}{=} (e^{ix})^n = e^{n(ix)} = e^{i(nx)} \underset{\text{E's form.}}{=} \cos nx + i\sin nx$

Let $n=2$, then we have:

$$\cos 2x + i\sin 2x = (\cos x + i\sin x)^2$$
$$= \cos^2 x + i^2\sin^2 x + 2i\sin x\cos x$$
$$(\cos 2x) + i(\sin 2x) = (\cos^2 x - \sin^2 x) + i(2\sin x\cos x)$$

Two complex numbers are equal if and only if their real and imaginary parts are equal respectively, thus:

$\cos 2x = \cos^2 x - \sin^2 x$ (from real parts' equality)

$\sin 2x = 2\sin x\cos x$ (from imaginary parts' equal.)

(9.) $t^2\dfrac{d^2y}{dt^2} + 5t\dfrac{dy}{dt} - 5y = 0$

Let $y(t) = t^\lambda \Rightarrow \dfrac{dy}{dt} = \lambda t^{\lambda-1} \Rightarrow \dfrac{d^2y}{dt^2} = \lambda(\lambda-1)t^{\lambda-2}$

$t^2\lambda(\lambda-1)t^{\lambda-2} + 5t\lambda t^{\lambda-1} - 5t^\lambda = 0$

$(\lambda^2-\lambda)t^\lambda + 5\lambda t^\lambda - 5t^\lambda = 0$ / Assume $t \neq 0$ divide by t^λ

$\lambda^2 - \lambda + 5\lambda - 5 = 0$

$\lambda^2 + 4\lambda - 5 = 0$

$\hookrightarrow \lambda_{1,2} = \dfrac{-4 \pm \sqrt{4^2 - 4(-5)}}{2} = \dfrac{-4 \pm \sqrt{36}}{2} = \dfrac{-4 \pm 6}{2}$

$\lambda_1 = \dfrac{-4+6}{2} = 1, \quad \lambda_2 = \dfrac{-4-6}{2} = -5$

$\Downarrow$

if $t=0 \Rightarrow y=0$

$y=0$ is clearly a singular soln.

Two lin. independent solns:

$\boxed{y_1(t) = t^1 = t}$ and $\boxed{y_2(t) = t^{-5} = \frac{1}{t^5}}$