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MATH 527 501-03.
HW4 Solns.
(Applications)

① $\frac{dT}{dt} = k(T - T_m)$ Newton's Law of Cooling

$T_m = 10°F$

$T(0) = 70°F$

$T(30) = 50°F$

$T(60) = ?$

For $T = 15°F$, $t = ?$

$\frac{dT}{dt} = k(T-10)$, separable

$\frac{1}{T-10}\frac{dT}{dt} = k$

$\frac{d}{dt}\left(\ln|T-10|\right) = k$

Integrate both sides wrt t

$\ln|T-10| = kt + C$

$|T-10| = e^{kt+C}$

$T - 10 = Ce^{kt}$

$\boxed{T(t) = 10 + Ce^{kt}}$

$T(0) = 10 + Ce^0$

$70 = 10 + C$

$\boxed{60 = C}$

$T(30) = 10 + 60e^{k(30)}$

$50 = 10 + 60e^{k(30)}$

$\frac{40}{60} = e^{k30}$

$\frac{1}{30}\ln\left(\frac{2}{3}\right) = k$

$-0.013516 = k$

$\boxed{T(t) = 10 + 60e^{-0.013516t} \text{ for } t \text{ in seconds, } T \text{ in Fahrenheit}}$

$\boxed{T(60) = 36.7°F}$

$15 = 10 + 60e^{-0.013516t}$

$\frac{\ln\left(\frac{5}{60}\right)}{-0.013516} = t$

$\rightarrow$ $\boxed{t = 183.85 \text{ secs for } T = 15°F}$ 3.05 min

2) Mixtures $\frac{dA}{dt}$ = input rate of salt − output rate of salt

$$\frac{dA}{dt} = R_{in} - R_{out}$$

, $A(t)$ = amount of salt in the soln at time t in grams

$$R_{in} = 1\,\frac{g}{L} \cdot 4\,\frac{L}{min} = 4\,\frac{g}{min}$$

$$R_{out} = \frac{A(t)}{200}\,\frac{g}{L} \cdot 4\,\frac{L}{min}$$

$$\frac{dA}{dt} = 4 - \frac{A}{50}$$

$$\boxed{\frac{dA}{dt} + \frac{A}{50} = 4}$$

+5

$$\mu(t) = e^{\int \frac{1}{50}\,dt} = e^{\frac{1}{50}t}$$

$$e^{\frac{1}{50}t}\left(\frac{dA}{dt} + \frac{A}{50} = 4\right)$$

$$\frac{d}{dt}\left(e^{\frac{1}{50}t}A\right) = 4e^{\frac{1}{50}t}$$

Int wrt t.

$$e^{\frac{1}{50}t}A = \frac{4}{\frac{1}{50}}e^{\frac{1}{50}t} + C$$

$$A(t) = e^{-\frac{1}{50}t}\left(200e^{\frac{1}{50}t} + C\right)$$

$$\boxed{A(t) = 200 + Ce^{-\frac{1}{50}t}}$$

+5

$\boxed{\text{use IC } A(0) = 30\text{ g.}}$ +5

$$A(0) = 200 + Ce^{-\frac{1}{50}(0)}$$

$$30 = 200 + C$$

$$\boxed{-170 = C}$$

+5

Then

$$\boxed{A(t) = 200 - 170e^{-\frac{1}{50}t}}$$

+5

3) Second-Order Chemical Reaction

rate of production of C = $X(t)$.
at time t

$$\frac{dX}{dt} \propto \begin{pmatrix}\text{amt of A remaining} \\ \text{at time } t\end{pmatrix}\begin{pmatrix}\text{amt of B remaining} \\ \text{at time } t\end{pmatrix}$$

2g of A + 1g of B $\rightarrow$ 3g of C

Then for each gram of B, 2g of A is used.

For a grams of A, b grams of B, we have

$a + b = X$ grams of C. $a = 2b$ and

$2b + b = X$

then $b = \frac{X}{3}$ and $a = 2b = \frac{2}{3}X$

at $t = 0$, there are 100g of A, 50 g of B, 0 g of C.

Then $\frac{dX}{dt} \propto \left(100 - \frac{2}{3}x\right)\left(50 - \frac{1}{3}X\right)$

i.e. $\frac{dX}{dt} = k_1(300 - 2x)(150 - X)$ which is separable.

$$\frac{dX}{dt} = k_1 \, 2(150 - X)(150 - X)$$

$$\boxed{\frac{dX}{dt} = k_2 (150 - X)^2}$$ +5

$$\frac{1}{(150-x)^2}\frac{dX}{dt} = k_2$$

$$\left(\int \frac{1}{(150-x)^2}\,dx = \int -u^{-2}\,du = \frac{-u^{-1}}{-1} = u^{-1} \right)$$

$u = 150 - x$
$du = -dx$

$$\frac{d}{dt}\left(\frac{1}{150-x}\right) = k_2$$

Integrate wrt t.

$$\frac{1}{150-x} = k_2 t + C$$

$$\frac{1}{kt+C} = 150 - X$$

+5 $$\boxed{X(t) = 150 - \frac{1}{k_2 t + C}}$$ Drop subscript in k_2 $(k_2 = k)$

Use ICs ① $X(10) = 10g$.

$$X(10) = 150 - \frac{1}{k(10)+C} = 10$$ used ICs correctly +5

$$140 = \frac{1}{10k + C}$$

$$10k + C = \frac{1}{140}$$

$$C = \frac{1}{140} - 10k$$

② $X(0) = 0g$.

$$X(0) = 150 - \frac{1}{C} = 0$$

$$150 = \frac{1}{C} \rightarrow \boxed{C = \frac{1}{150}}$$

Then $C = \frac{1}{140} - 10k$ and $C = \frac{1}{150}$

so $\frac{1}{150} = \frac{1}{140} - 10k$

$$\frac{\frac{1}{150} - \frac{1}{140}}{-10} = k.$$

$$\boxed{k = 4.7619e^{-5}}$$

So finally, $X(t) = 150 - \frac{1}{\left(4.7619e^{-5}\, t + \left(\frac{1}{150}\right)\right)}$

i.e. $\boxed{X(t) = 150 - \frac{150}{0.0071429\,t + 1}}$ $\left(\begin{array}{l} 4.7619e^{-5} \cdot 150 \\ = 7.1429e^{-3} \end{array}\right)$

As $t \to \infty$, $X(t) \to \lim_{t \to \infty} 150 - \frac{150}{0.0071429t + 1} = 150.$

$\boxed{\text{As } t \to \infty,\ X(t) \to 150.}$ +5

For $X(t) = \frac{1}{2} 150 = 75$, $t = ?$

$$75 = 150 - \frac{150}{0.0071429t + 1}$$

$$-75\,(0.0071429t + 1) = -150$$

$$-75\,(0.0071429t) = -150 + 75$$

$$t = \frac{-75}{-75\,(0.0071429)} = 140$$

$\boxed{\begin{array}{l} \text{For } X = 75 \\ t = 140 \text{ min} \end{array}}$ +5