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MATH 527 Sect. 1-3
Fall 2015
HW #1 Solns.

#1-5 Separation of Variables

1) $\frac{dy}{dt} = (1+t)(1+y)$

$\frac{1}{1+y}\frac{dy}{dt} = 1+t$ for $y \neq -1$

$\frac{d}{dt}\left(\ln|1+y|\right) = 1+t$

Integrate wrt t to both sides of the eqn.

$\int \frac{d}{dt}\left(\ln|1+y|\right) dt = \int 1+t \, dt$

$\ln|1+y| + C_1 = t + \frac{t^2}{2} + C_2$

Solve for y and combine constants.

$|1+y| = e^{t+\frac{t^2}{2}+C_2-C_1}$

$1+y = \pm e^{t+\frac{t^2}{2}+C_2-C_1}$

$y = -1 \pm e^t e^{\frac{t^2}{2}} C$

$\pm$ is collapsed into constant

$\boxed{y(t) = -1 + Ce^t e^{\frac{t^2}{2}}}$ (for $y \neq -1$)

$\boxed{y(t) = -1 \text{ is a singular soln.}}$

check soln

$\frac{d}{dt}(y(t)) = C\left[e^t e^{\frac{t^2}{2}}\left(\frac{1}{2}2t\right) + e^t e^{\frac{t^2}{2}}\right]$

$= Ce^t e^{\frac{t^2}{2}}(t+1)$

and $Ce^t e^{\frac{t^2}{2}}$ is in fact $y(t)+1$

$\frac{d}{dt}(y(t)) = (y(t)+1)(t+1)$ ✓

2) $\frac{dy}{dt} = 1 - t + y^2 - ty^2$

rewrite in standard form

$$\frac{dy}{dt} = 1 - t + y^2(1-t)$$

$$\frac{dy}{dt} = (1-t)(1+y^2) \leftarrow \text{standard form!}$$

$$\frac{1}{1+y^2}\frac{dy}{dt} = 1 - t$$

$$\frac{d}{dt}\left(\tan^{-1}(y)\right) = 1-t$$

Integrate wrt t to both sides of eqn.

$$\int \frac{d}{dt}\left(\tan^{-1}(y)\right) dt = \int 1-t \, dt$$

$$\tan^{-1}(y) + C_1 = t - \frac{t^2}{2} + C_2$$

Combine constants and solve for y(t).

$$\boxed{y(t) = \tan\left(t - \frac{t^2}{2} + C\right)}$$

3) $\frac{dy}{dx} = e^{x+y+3}$

rewrite in standard form

$$\frac{dy}{dx} = e^x e^y e^3$$

$$e^{-y}\frac{dy}{dx} = e^x e^3$$

$$\frac{d}{dx}\left(\frac{e^{-y}}{-1}\right) = e^x e^3$$

Integrate both sides of the eqn wrt to x.

$$\int \frac{d}{dx}\left(-e^{-y}\right) dx = \int e^x e^3 \, dx$$

$$-e^{-y} + C_1 = e^3 e^x + C_2$$

3) continued...

$$-e^{-y}+C_1=e^3e^x+C_2$$

Combine constants and solve for $y(x)$.

$$e^{-y}=C-e^3e^x$$

$$-y=\ln\left(C-e^3e^x\right)$$

$\boxed{y(x)=-\ln\left(C-e^3e^x\right)}$ or $\boxed{y(x)=\ln\left(\left(C-e^3e^x\right)^{-1}\right)}$

4) $$\frac{dy}{dx}+2xy^2=0$$

Rewrite in standard form for separation of variables.

$$\frac{dy}{dx}=-2xy^2$$

$$y^{-2}\frac{dy}{dx}=-2x \quad \text{for } y\neq 0$$

$$\frac{d}{dx}\left(-y^{-1}\right)=-2x$$

Integrate both sides of the eqn wrt x.

$$\int\frac{d}{dx}\left(-y^{-1}\right)dx=\int-2x\,dx$$

$$-y^{-1}+C_1=\frac{-2x^2}{2}+C_2$$

Combine constants & solve for $y(x)$.

$$-y^{-1}=-x^2+C$$

$$\frac{1}{y}=x^2+C \quad (-C\to C)$$

$\boxed{y(x)=\frac{1}{x^2+C}}$ for $y\neq 0$

$\boxed{y(x)=0 \text{ is a singular soln.}}$

5) $\frac{dy}{dt} = \frac{2t}{y+yt^2}$; $y(2)=3$

* partial credit within each group of points, up to you.

Rewrite in standard form.

$$\frac{dy}{dt} = \frac{2t}{y(1+t^2)}$$

$$y\frac{dy}{dt} = \frac{2t}{1+t^2}$$

$$\frac{d}{dt}\left(\frac{y^2}{2}\right) = \frac{2t}{1+t^2}$$

Integrate Both sides of eqn wrt t.

$$\int \frac{d}{dt}\left(\frac{y^2}{2}\right)dt = \int \frac{2t}{1+t^2}\,dt \longrightarrow \begin{array}{l} u = 1+t^2 \\ du = 2t\,dt \end{array} \quad \int \frac{du}{u}$$

$$\frac{y^2}{2} + C_1 = \ln|1+t^2| + C_2$$ → +10 correct integration.

Combine constants & solve for $y(t)$.

$$y^2 = 2\left(\ln|1+t^2| + C\right)$$

$$\boxed{y(t) = \pm\sqrt{2\ln|1+t^2| + C}}$$ $(2C \to C)$ → +5 solve for $y(t)$ in terms of C

Initial value $y(2)=3$ means when $t=2$, $y=3$.

$$y(2) = \pm\sqrt{2\ln|1+2^2| + C}$$

$$3 = \pm\sqrt{2\ln(5) + C}$$ → 3 can't be $-\sqrt{\quad}$, so that soln is eliminated.

$$9 = 2\ln(5) + C$$

$$\boxed{9 - 2\ln(5) = C}$$

then we have

$$\boxed{y(t) = +\sqrt{2\ln|1+t^2| + 9 - 2\ln(5)}}$$ → +5 final answer with correct value of C.

+5 organized and clear

6–9 using an integrating factor.

6.) $\frac{dy}{dt} + y\cos t = 0$ Check: no singular solns.

$$\mu(t) = e^{\int P\,dt} = e^{\int \cos(t)\,dt} = e^{\sin(t) + \cancel{C}}$$

+5 Correct calculation and usage of $\mu(t)$.

Multiply Both sides of equ by $\mu(t)$.

$$e^{\sin(t)}\left(\frac{dy}{dt} + y\cos(t) = 0\right)$$

$$e^{\sin(t)}\frac{dy}{dt} + e^{\sin(t)}y\cos(t) = 0$$

LHS is the result of a product rule derivative

$$\frac{d}{dt}\left(y e^{\sin(t)}\right) = 0$$

Integrate Both sides of equ wrt t.

$$\int \frac{d}{dt}\left(y e^{\sin(t)}\right)dt = \int 0\,dt$$

+10 correct integration

$$y e^{\sin(t)} + C_1 = C_2$$

$$\boxed{y(t) = Ce^{-\sin(t)}}$$

+5 final answer

+5 organized and clear (all work shown)

7) $\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{1+x^2}$

check: no singular soln.

$$P(x) = \frac{2x}{1+x^2}$$

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln|1+x^2| + \cancel{c}^{0}} = |1+x^2| = 1+x^2$$

Multiply by $\mu(x)$.

$$(1+x^2)\left(\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{1+x^2}\right)$$

$$(1+x^2)\frac{dy}{dx} + 2xy = 1$$

LHS is the result of a product rule derivative.

$$\frac{d}{dx}\left(y(1+x^2)\right) = 1$$

Integrate Both sides wrt x.

$$\int \frac{d}{dx}\left(y(1+x^2)\right)dx = \int 1\,dx$$

$$y(1+x^2) + C_1 = x + C_2$$

Combine constants and solve for $y(x)$.

$$\boxed{y(x) = \frac{x+C}{1+x^2}}$$

8) $(1+t^2)\frac{dy}{dt} + ty = (1+t^2)^{\frac{5}{2}}$

$$\frac{dy}{dt} + \frac{t}{1+t^2}y = (1+t^2)^{\frac{3}{2}}$$ check, no singular solns

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{t}{1+t^2}dt} = e^{(\ln|1+t^2|)\frac{1}{2}} = e^{\ln(\sqrt{|1+t^2|})} = \sqrt{1+t^2}$$

Multiply Both sides by $\mu(t)$.

$$\sqrt{1+t^2}\left(\frac{dy}{dt} + \frac{t}{1+t^2}y = (1+t^2)^{\frac{3}{2}}\right)$$

$$(1+t^2)^{\frac{1}{2}}\frac{dy}{dt} + t(1+t^2)^{-\frac{1}{2}}y = (1+t^2)^2$$

LHS is the result of a product rule derivative

$$\frac{d}{dt}\left((1+t^2)^{\frac{1}{2}}y\right) = (1+t^2)^2$$

Integrate both sides wrt t

$$\int \frac{d}{dt}\left((1+t^2)^{\frac{1}{2}}y\right)dt = \int 1+2t^2+t^4\,dt$$

$$y(1+t^2)^{\frac{1}{2}} + C_1 = t + \frac{2t^3}{3} + \frac{t^5}{5} + C_2$$

Combine constants & solve for $y(t)$.

$$\boxed{y(t) = \left(t + \frac{2}{3}t^3 + \frac{t^5}{5} + C\right)\left(1+t^2\right)^{-\frac{1}{2}}}$$

9) $\frac{dy}{dx} - 2xy = x$, $y(0) = 1$ Check no singular solns.

$$\mu(x) = e^{\int P(x)dx} = e^{-\int 2x\,dx} = e^{-2\frac{x^2}{2} + \cancel{C}^{0}} = e^{-x^2}$$

Multiply Both sides by $\mu(x)$.

$$e^{-x^2}\left(\frac{dy}{dx} - 2xy = x\right)$$

$$e^{-x^2}\frac{dy}{dx} - 2xye^{-x^2} = e^{-x^2}x$$

LHS is the result of a product rule derivative.

$$\frac{d}{dx}\left(e^{-x^2}y\right) = e^{-x^2}x$$

Integrate wrt x. for both sides of eqn.

$$\int \frac{d}{dx}\left(e^{-x^2}y\right)dx = \int e^{-x^2}x\,dx \quad \begin{array}{l} u = -x^2 \\ du = -2x\,dx \end{array} \quad -\frac{1}{2}\int e^{u}\,du$$

$$e^{-x^2}y + C_2 = -\frac{1}{2}e^{-x^2} + C_1$$

Combine constants & solve for $y(x)$.

$$y(x) = \left(-\frac{1}{2}e^{-x^2} + C\right)e^{x^2}$$

$$\boxed{y(x) = -\frac{1}{2} + Ce^{x^2}}$$

Initial value $y(0) = 1$ when $x = 0$, $y = 1$

$$y(0) = -\frac{1}{2} + Ce^{0^2}$$

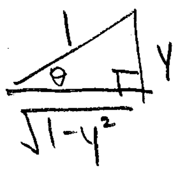
$$1 = -\frac{1}{2} + C$$

$$\boxed{\frac{3}{2} = C} \longrightarrow \boxed{y(x) = -\frac{1}{2} + \frac{3}{2}e^{x^2}}$$

10) $\dfrac{dy}{dx} = x\sqrt{1-y^2}$

Separation of variables.

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = x \longrightarrow y \neq \pm 1$$



$$y = \sin\theta$$
$$dy = \cos\theta\, d\theta$$
$$\sqrt{1-y^2} = \cos\theta$$

$$\frac{1}{\cos\theta}\cos\theta\frac{d\theta}{dx} = x$$

$$\frac{d}{dx}(\theta) = x$$

$$\frac{d}{dx}\left(\sin^{-1}(y)\right) = x$$

Integrate both sides of eqn wrt x.

$$\int \frac{d}{dx}\left(\sin^{-1}(y)\right)dx = \int x\, dx$$

$$\sin^{-1}(y) + C_2 = \frac{x^2}{2} + C_1$$

Combine constants & solve for $y(x)$.

$$\boxed{y(x) = \sin\left(\frac{x^2}{2} + C\right) \quad \text{for } y \neq \pm 1}$$

$$\boxed{y(x) = \pm 1 \quad \text{are singular solns.}}$$