David Miller
math527.7-9
10/15/15

Math 527
Homework 6

Pt 1

Questions #6, 7 are targeted
Typical grading for #1–5

1.) $y''+3y=x^3-1 \Rightarrow y_p=Ax^3+Bx^2+Cx+D$

Find y_h/y_c, the complimentary or homogeneous solution:

$y''+3y=0$

$r^2+3=0 \Rightarrow r=\pm i\sqrt{3}$

Thus, $y_h = A\cos\sqrt{3}x + B\sin\sqrt{3}x$

Find y_p, the particular solution:

$y_p = Ax^3+Bx^2+Cx+D$

$y_p' = 3Ax^2+2Bx+C$

$y_p'' = 6Ax+2B$ Plug back into differential equation and solve for A, B, C, D

$6Ax+2B+3Ax^3+3Bx^2+3Cx+3D = x^3-1$

$\Rightarrow 3Ax^3 = x^3 \Rightarrow 3A=1 \Rightarrow A=\frac{1}{3}$

$\Rightarrow 3Bx^2=0 \Rightarrow B=0$

$\Rightarrow 6Ax+3Cx=0 \Rightarrow 3C=-6A \Rightarrow 3C=-2 \Rightarrow C=-\frac{2}{3}$

$\Rightarrow 2B+3D=-1 \Rightarrow 3D=-1 \Rightarrow D=-\frac{1}{3}$

Thus, $y_p = \frac{1}{3}x^3 - \frac{2}{3}x - \frac{1}{3}$

Our general solution to the differential equation is

$$y(x) = A\cos\sqrt{3}x + B\sin\sqrt{3}x + \frac{1}{3}x^3 - \frac{2}{3}x - \frac{1}{3}$$

2.) $y''+4y'+4y = te^{2t} \Rightarrow y_p=(At+B)Ce^{2t}$, more simply, $Ate^{2t}+Be^{2t}$

Find y_h: $y''+4y'+4y=0$

$r^2+4r+4=0$

$(r+2)(r+2)=(r+2)^2=0 \Rightarrow r+2=0 \Rightarrow r=-2$ repeated real root

Thus, $y_h = c_1e^{-2t}+c_2te^{-2t}$

Find y_p: $y_p = Ate^{2t}+Be^{2t} \Rightarrow y_p' = Ae^{2t}+2Ate^{2t}+2Be^{2t} \Rightarrow y_p'' = 2Ae^{2t}+2Ae^{2t}+4Ate^{2t}+4Be^{2t}$

$4Ate^{2t}+4Ae^{2t}+4Be^{2t}+4Ae^{2t}+8Ate^{2t}+8Be^{2t}+4Ate^{2t}+4Be^{2t} = te^{2t}$

$16Ate^{2t} = te^{2t} \Rightarrow 16A=1 \Rightarrow A=\frac{1}{16}$

$8Ae^{2t}+16Be^{2t}=0 \Rightarrow -8A=16B \Rightarrow -\frac{1}{2}=16B \Rightarrow B=-\frac{1}{32}$

Thus, $y_p = \frac{1}{16}te^{2t}-\frac{1}{32}e^{2t}$

Therefore, $y(t) = c_1e^{-2t}+c_2te^{-2t}+\frac{1}{16}te^{2t}-\frac{1}{32}e^{2t}$

3.) $y'' + 2y' + y = e^{-t} \Rightarrow y_p = Ae^{-t}$

Find y_h: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$ repeated real root

Thus, $y_h = c_1 e^{-t} + c_2 t e^{-t}$

Note: $y_p = Ae^{-t}$ is not linearly independent of y_h, nor is Ate^{-t}, thus

$y_p = At^2 e^{-t}$.

$y_p' = 2Ate^{-t} - At^2 e^{-t} \Rightarrow y_p'' = 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2 e^{-t}$

$\cancel{At^2 e^{-t}} - \cancel{4Ate^{-t}} + 2Ae^{-t} + \cancel{4Ate^{-t}} - 2\cancel{At^2 e^{-t}} + \cancel{At^2 e^{-t}} = e^{-t}$

$\Rightarrow 2Ae^{-t} = e^{-t} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

Thus, $y_p = \frac{1}{2} t^2 e^{-t}$

Therefore, $y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}$

4.) $y'' + 4y = t\sin 2t \Rightarrow y_p = (At+B)(C\cos 2t + D\sin 2t)$, more simply, $At\cos 2t + Bt\sin 2t + C\cos 2t + D\sin 2t$

Find y_h: $r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$

Thus, $y_h = U\cos 2t + V\sin 2t$

Note: $y_p = At\cos 2t + Bt\sin 2t + C\cos 2t + D\sin 2t$ is not linearly independent of y_h, thus $y_p = At^2\cos 2t + Bt^2\sin 2t + Ct\cos 2t + Dt\sin 2t$.

$y_p' = 2At\cos 2t - 2At^2\sin 2t + 2Bt\sin 2t + 2Bt^2\cos 2t + C\cos 2t - 2Ct\sin 2t + D\sin 2t + 2Dt\cos 2t$

$y_p'' = [2A\cos 2t - 4At\sin 2t - 4At\sin 2t - 4At^2\cos 2t + 2B\sin 2t + 4Bt\cos 2t + 4Bt\cos 2t - 4Bt^2\sin 2t$

$-2C\sin 2t - 2C\sin 2t - 4Ct\cos 2t + 2D\cos 2t + 2D\cos 2t - 4Dt\sin 2t]$

$[\cancel{-4At^2\cos 2t} - \cancel{4Bt^2\sin 2t} + (8B-4C)t\cos 2t + (-8A-4D)t\sin 2t + (2A+4D)\cos 2t + (2B-4C)\sin 2t$

$+ \cancel{4At^2\cos 2t} + \cancel{4Bt^2\sin 2t} + 4Ct\cos 2t + 4Dt\sin 2t] = t\sin 2t$

$\Rightarrow 8B - 4C + 4C = 0 \Rightarrow B = 0$

$\Rightarrow -8A - 4D + 4D = 1 \Rightarrow A = -\frac{1}{8}$

$\Rightarrow 2A + 4D = 0 \Rightarrow 4D = \frac{1}{4} \Rightarrow D = \frac{1}{16}$

$\Rightarrow 2B - 4C = 0 \Rightarrow C = 0$

Thus, $y_p = -\frac{1}{8} t^2\cos 2t + \frac{1}{16} t\sin 2t$

Therefore, $y(t) = U\cos 2t + V\sin 2t - \frac{1}{8} t^2\cos 2t + \frac{1}{16} t\sin 2t$

Half-angle formula $\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$

5.) $y'' - 2y' + 5y = 2\cos^2 x \Rightarrow y_p = A + B\cos 2x + C\sin 2x$

Find y_h: $r^2 - 2r + 5 = 0 \Rightarrow a = 1,\ b = -2,\ c = 5$

$r = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} \Rightarrow r = 1 \pm 2i$

Thus, $y_h = e^x (c_1 \cos 2x + c_2 \sin 2x)$

Find y_p: $y_p = A + B\cos 2x + C\sin 2x$

$y_p' = -2B\sin 2x + 2C\cos 2x$

$y_p'' = -4B\cos 2x - 4C\sin 2x$

$-4B\cos 2x - 4C\sin 2x + 4B\sin 2x - 4C\cos 2x + 5A + 5B\cos 2x + 5C\sin 2x = 1 + \cos 2x$

$-4B - 4C + 5B = 1 \Rightarrow B = 1 + 4C$

$5A = 1 \Rightarrow A = \frac{1}{5}$

$-4C + 4B + 5C = 0 \Rightarrow C = -4B \Rightarrow C = -4(1+4C) \Rightarrow 17C = -4 \Rightarrow C = \frac{-4}{17}$

$\Rightarrow B = 1 + 4\left(-\frac{4}{17}\right) = \frac{1}{17}$

Thus, $y_p = \frac{1}{17}\cos 2x - \frac{4}{17}\sin 2x + \frac{1}{5}$

Therefore, $y(x) = e^x(c_1\cos 2x + c_2 \sin 2x) + \frac{1}{17}\cos 2x - \frac{4}{17}\sin 2x + \frac{1}{5}$

6.) $y'' + y' - 6y = \sin t + te^{2t} \Rightarrow y_p = A\cos t + B\sin t + Cte^{2t} + De^{2t}$ 25 total points

Find y_h: $r^2 + r - 6 = 0 \Rightarrow (r-2)(r+3) = 0 \Rightarrow r = 2, -3$

Thus, $y_h = c_1 e^{-3t} + c_2 e^{2t}$ +5

Note: y_p is not linearly independent of y_h, thus

$y_p = A\cos t + B\sin t + Ct^2e^{2t} + Dte^{2t}$ +5

$y_p' = -A\sin t + B\cos t + 2Cte^{2t} + 2Ct^2e^{2t} + De^{2t} + 2Dte^{2t}$

$y_p'' = -A\cos t - B\sin t + 2Ce^{2t} + 4Cte^{2t} + 4Cte^{2t} + 4Ct^2e^{2t} + 2De^{2t} + 2De^{2t} + 4Dte^{2t}$

$[-A\cos t - B\sin t + (2C+4D)e^{2t} + (8C+4D)te^{2t} + 4Ct^2e^{2t} - A\sin t + B\cos t + 2Cte^{2t}$

$+ 2Ct^2e^{2t} + De^{2t} + 2Dte^{2t} - 6A\cos t - 6B\sin t - 6Ct^2e^{2t} - 6Dte^{2t}] = \sin t + te^{2t}$

$-A + B - 6A = 0 \Rightarrow 7A = B$

$-B - A - 6B = 1 \Rightarrow +50A = 1 \Rightarrow A = -\frac{1}{50} \Rightarrow B = -\frac{7}{50}$ +5 for A, B, C, D

$2C + 4D + D = 0 \Rightarrow C = -\frac{5}{2}D$

$8C + 4D + 2C + 2D - 6D = 1 \Rightarrow 10C = 1 \Rightarrow C = \frac{1}{10} \Rightarrow D = -\frac{1}{25}$

Thus, $y_p = -\frac{1}{50}\cos t - \frac{7}{50}\sin t + \frac{1}{10}t^2e^{2t} - \frac{1}{25}te^{2t}$ +5

Therefore, $y(t) = c_1 e^{-3t} + c_2 e^{2t} + \frac{1}{10}t^2e^{2t} - \frac{1}{25}te^{2t} - \frac{7}{50}\sin t - \frac{1}{50}\cos t$ +5

Pt 4

Recall $\cosh x + \sinh x = e^{x}$
$\cosh x - \sinh x = e^{-x}$
$\cosh^2(x) + \sinh^2(x) = 1$

7.) $y'' - y = \cosh(x)$, $y(0) = 2$, $y'(0) = 12$ $\Rightarrow y_p = A\cosh(x) + B\sinh(x)$ 25 total points

Find y_h: $r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = \pm 1$

$y_h = c_1 e^{-x} + c_2 e^{x}$ ← +5

Find y_p: $y_p = A\cosh(x) \pm B\sinh(x)$, more simply, $Ae^{x} + Be^{-x}$

Note: y_p is not linearly independent of y_h, thus $y_p = Axe^{x} + Bxe^{-x}$ ← +5

$y_p' = Ae^{x} + Axe^{x} + Be^{-x} - Bxe^{-x}$

$y_p'' = Ae^{x} + Ae^{x} + Axe^{x} - Be^{-x} - Be^{-x} + Bxe^{-x}$

$2Ae^{x} + \cancel{Axe^{x}} - 2Be^{-x} + \cancel{Bxe^{-x}} - \cancel{Axe^{x}} - \cancel{Bxe^{-x}} = \cosh(x) = \frac{1}{2}e^{-x} + \frac{1}{2}e^{x}$

$2A = \frac{1}{2} \Rightarrow A = \frac{1}{4}$

$-2B = \frac{1}{2} \Rightarrow B = -\frac{1}{4}$ > +5

Thus, $y_p = \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x}$

Therefore, $y(x) = c_1 e^{-x} + c_2 e^{x} + \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x}$

$y'(x) = -c_1 e^{-x} + c_2 e^{x} + \frac{1}{4}e^{x} + \frac{1}{4}xe^{x} - \frac{1}{4}e^{-x} + \frac{1}{4}xe^{-x}$

Initial conditions: $y(0) = 2 = c_1 + c_2 \Rightarrow c_1 = 2 - c_2 \Rightarrow c_1 = -5$ ← +5

$y'(0) = 12 = -c_1 + c_2 + \frac{1}{4} - \frac{1}{4} \Rightarrow 12 = -2 + c_2 + c_2 \Rightarrow c_2 = 7$

Thus, $y(x) = -5e^{-x} + 7e^{x} + \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x}$ ← +5

$= \frac{1}{4}e^{-x}\left(xe^{2x} + 28e^{2x} - x - 20\right)$

$= \frac{1}{4}e^{-x}\left(e^{2x}(x+28) - x - 20\right)$