

Homework #2
Due Wed, Oct 17th in lecture

IAM 961, UNH fall 2012

1. In this exercise you will verify numerically how uniqueness of the SVD breaks down when two singular values are equal. Let A be an $m \times m$ real-valued matrix with singular value decomposition $A = UDV^*$. Suppose two (and only two) singular values are equal, $d_n = d_{n+1}$, so that the SVD is not unique, and let $A = WEZ^*$ be a different SVD of A .

(a) What can you say about the relations of D and E ? U and W ? V and Z ? In particular, what is the relationship between the left singular vectors $\{u_n, u_{n+1}\}$ of the first SVD and the left singular vectors $\{w_n, w_{n+1}\}$ of the second SVD? What is the similar relationship between $\{v_n, v_{n+1}\}$ and $\{z_n, z_{n+1}\}$? Please try to answer this question as specifically as you can before moving on to (b).

(b) Verify your answer to (a) numerically as follows. Construct a random $M \times M$ matrix A with two equal singular values and two different singular value decompositions, using following Matlab code to get you started.

```
M = 10; % or some other number
[U, R] = qr(randn(M)); % construct a random unitary U
[V, R] = qr(randn(M)); % ditto V
d = 10.^(4*(rand(M,1)-0.5)) % logarithmically distributed singular values

n = 4; % or some other number between 1 and N-1
d(n) = d(n+1);
D = diag(d);

A = U*D*V'; % A = U D V^* is an SVD of A

[W,E,Z] = svd(A); % A = W E Z^* is another SVD of A
```

First verify that $A = UDV^* = WEZ^*$ by evaluating $\text{norm}(A - U*D*V')$ and $\text{norm}(A - W*E*Z')$. Similarly, check that U, V, W, Z are unitary by evaluating $\text{norm}(U'*U - \text{eye}(m))$, etc. Next, the key answer to (a) (in this context, where $n = 4$) is that $\text{span}(u_4, u_5) = \text{span}(w_4, w_5)$ and $\text{span}(v_4, v_5) = \text{span}(z_4, z_5)$. Devise a way of verifying these relations numerically, and perform the necessary computations to verify for the specific vectors generated by the above code. You should be able to do it with eight very similar one-line Matlab expressions that each evaluate to zero (or approximately zero) if the spans are equal.

(c) What's the answer to (a) if we we had *three* equal singular values $d_4 = d_5 = d_6$, and what expressions would you evaluate to verify these relations numerically?

2. In this exercise you will verify numerically how uniqueness works for the SVD of a *complex* matrix with *distinct* singular values. Let A be a complex-valued $m \times m$ matrix with singular value decomposition $A = UDV^*$ and distinct singular values $d_1 > d_2 > \dots > d_{m-1} > d_m$. Let $A = WEZ^*$ be another SVD of A .

(a) What can you say about the relations of D and E ? U and W ? V and Z ? In particular, what is the relationship between a left singular vector u_n of the first SVD and the corresponding left singular vector w_n of the second SVD? What is the similar relationship between v_n and z_n ?

(b) Verify your answer to (a) numerically as follows. Construct a random $M \times M$ complex matrix A with distinct singular values and two different singular value decompositions, using following Matlab code to get you started.

```
M = 10; % or some other number
[U, R] = qr(randn(M) + i*rand(M)); % U = random complex unitary matrix
[V, R] = qr(randn(M) + i*rand(M)); % ditto for V
d = 10.^(4*(rand(M,1)-0.5)); % log-distributed singular values

D = diag(d);

A = U*D*V'; % A = U D V^* is an SVD of A

[W,E,Z] = svd(A); % A = W E Z^* is another SVD of A
```

Verify that $A = UDV^* = WEZ^*$ by evaluating $\text{norm}(A - U*D*V')$ and $\text{norm}(A - W*E*Z')$. Similarly, check that U, V, W, Z are unitary by evaluating $\text{norm}(U'*U - \text{eye}(m))$, etc.

Next, devise a way of verifying your answer to (a) numerically, and perform the necessary computations to verify the relation between u_n and w_n and between v_n and z_n for at least one value of n . You should be able to do this with four very similar one-line Matlab expressions that each evaluate to zero (or approximately zero) for each n . If you are extra-clever, two Matlab expressions per n will do.

Please turn in hand-written answers to 1a, 1c, and 2a and print-outs of your Matlab session for 1b and 2b. Use Matlab's `diary` function to save the session to a file, and edit out mistakes and other extraneous information before printing.