Math 527 fall 2011 UNH
HW #9 solutions
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1a) Solve the IVP

$$y''+y'+y = 1+e^{-t} \qquad y(0)=3,\ y'(0)=-5$$

apply Lapl trans

$$s^2Y(s) - s\overset{3}{y(0)} - \overset{-5}{y'(0)} + sY(s) - \overset{3}{y(0)} + Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$(s^2+s+1)Y(s) - 3s + 2 = \frac{1}{s} + \frac{1}{s+1}$$

$$Y(s) = \frac{1}{s(s^2+s+1)} + \frac{1}{(s+1)(s^2+s+1)} + 3\frac{s}{(s^2+s+1)} - \frac{2}{s^2+s+1}$$

partial fractions

1st term
$$\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1} \qquad \text{cover-up } (s=0) \Rightarrow A=1$$

$$1 = 1(s^2+s+1) + Bs^2 + Cs \quad \Rightarrow B=-1,\ C=-1$$

$$\frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{s+1}{s^2+s+1}$$

2nd term
$$\frac{1}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1} \qquad \text{cover-up } (s=-1) \Rightarrow A=1$$

$$1 = (s^2+s+1) + (Bs+C)(s+1)$$

$$0 = s^2+s + Bs^2 + (B+C)s + C \quad \Rightarrow \quad C=0,\ B=-1$$

$$\frac{1}{(s+1)(s^2+s+1)} = \frac{1}{s+1} - \frac{s}{s^2+s+1}$$

$$\Rightarrow Y(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1} + \frac{1}{s+1} - \frac{s}{s^2+s+1} + \frac{3s}{s^2+s+1} - \frac{2}{s^2+s+1}$$

$$= \frac{1}{s} + \frac{1}{s+1} + \frac{s-3}{s^2+s+1}$$

Inv Lapl trans

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s-3}{s^2+s+1}\right\}$$

$$y(t) = 1 + \mathcal{L}^{-1}\left\{\frac{1}{s}\Big|_{s\to s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{(s+1/2)-7/2}{\underbrace{s^2+s+1/4}_{(s+1/2)^2}+3/4}\right\}$$

$$= 1 + e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s-7/2}{s^2+3/4}\Big|_{s\to s+1/2}\right\}$$

$$= 1 + e^{-t}\cdot 1 + e^{-\frac{1}{2}t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3/4} - \frac{7}{2}\sqrt{4/3}\,\frac{\sqrt{3/4}}{s^2+3/4}\right\}$$

$$\boxed{y(t) = 1 + e^{-t} + e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t - \frac{7}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]}$$

1b) Solve the IVP

$$y'' + 2y' + y = 3\delta(t-1) \qquad y(0) = y'(0) = 0$$

$$s^2Y(s) + s\cancel{y(0)}^{0} + \cancel{y'(0)}^{0} + 2sY(s) + 2\cancel{y(0)}^{0} + Y(s) = 3e^{-s}$$

$$(s^2+2s+1)Y(s) = 3e^{-s}$$

$$Y(s) = 3e^{-s}\frac{1}{s^2+2s+1}$$

$$y(t) - \mathcal{L}^{-1}\{Y(s)\} = 3\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2+2s+1}\right\}$$

$$= 3\mathcal{U}(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right]_{t\to t-1}$$

$$= 3\mathcal{U}(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s+1}\right\}\right]_{t\to t-1}$$

$$= 3\mathcal{U}(t-1)\left[e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right]_{t\to t-1}$$

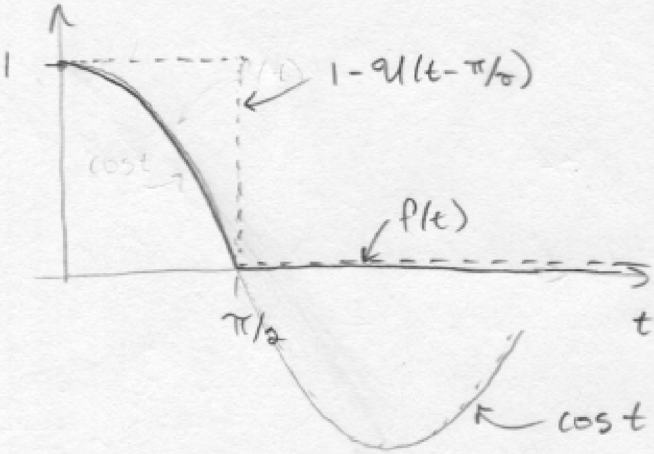
$$= 3\mathcal{U}(t-1)\left[e^{-t}t\right]_{t\to t-1}$$

$$\boxed{y(t) = 3\mathcal{U}(t-1)\,e^{-(t-1)}(t-1)}$$

1c) Solve the IVP

$$y'' + 4y = f(t) = \begin{cases} \cos t & 0 \le t < \pi/2 \\ 0 & \pi/2 \le t \end{cases}$$

$$y'(0) = y(0) = 0$$



$$1 - \mathcal{U}(t-\pi/2) = \begin{cases} 1 - 0 = 1 & 0 \le t < \pi/2 \\ 1 - 1 = 0 & \pi/2 \le t \end{cases}$$

So

$$f(t) = \left[1 - \mathcal{U}(t-\pi/2)\right] \cos t$$

$$y'' + 4y = \left[1 - \mathcal{U}(t-\pi/2)\right] \cos t$$

$$= \cos t - \mathcal{U}(t-\pi/2) \cos t$$

$$\cos t = -\sin(t - \pi/2)$$

$$y'' + 4y = \cos t + \mathcal{U}(t-\pi/2) \sin(t-\pi/2)$$

$\mathcal{L}$

$$s^2 Y(s) + s\cancel{y(0)}^{0} + \cancel{y'(0)}^{0} + 4Y(s) = \frac{s}{s^2+1} + \mathcal{L}\{\mathcal{U}(t-\pi/2)\sin(t-\pi/2)\}$$

$$(s^2+4)Y(s) = \frac{s}{s^2+1} + e^{-\pi/2\, s}\,\mathcal{L}\{\sin t\}$$

$$(s^2+4)Y(s) = \frac{s}{s^2+1} + e^{-\pi/2\, s}\frac{1}{s^2+1}$$

$$Y(s) = \frac{s}{(s^2+1)(s^2+4)} + e^{-\pi/2\, s}\frac{1}{(s^2+1)(s^2+4)}$$

Partial fractions

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+4)}$$

$$s = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$s = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + Cs + D$$

s^3 terms: $A = -C$, s terms: $1 = 4(-C) + C \Rightarrow C = -1/3,\ A = +1/3$

s^2 terms: $B = -D$, s^0 terms: $4B = -D \Rightarrow B = D = 0$

$$\therefore \quad \frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

(check on a few values of s, like $s=0$, $s=1$ (it works))

Divide by s to get

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}$$

So

$$Y(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{3}e^{-\pi/2 s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$y(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{e^{-\pi/2 s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)\right\}$$

$$= \frac{1}{3}\cos t - \frac{1}{3}\cos 2t + \frac{1}{3}\mathcal{U}(t-\pi/2)\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{2}\frac{2}{s^2+4}\right\}\right]_{t\to t-\pi/2}$$

$$= \frac{1}{3}\cos t - \frac{1}{3}\cos 2t + \frac{1}{3}\mathcal{U}(t-\pi/2)\left[\sin t - \frac{1}{2}\sin 2t\right]_{t\to t-\pi/2}$$

$$= \frac{1}{3}\left[\cos t - \cos 2t + \mathcal{U}(t-\pi/2)\left(\sin(t-\pi/2) - \frac{1}{2}\sin(2t-\pi)\right)\right]$$

$$\boxed{y(t) = \frac{1}{3}\left[\cos t - \cos 2t + \mathcal{U}(t-\pi/2)\left(\frac{1}{2}\sin 2t - \cos t\right)\right]}$$

2a) $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^n}\Big|_{s\to s-a}\right\}$

$= e^{at}\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\}$ let $n = m+1$

$= \frac{1}{m!} e^{at}\mathcal{L}^{-1}\left\{\frac{m!}{s^{m+1}}\right\}$

$= \frac{1}{m!} e^{at} t^m$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{1}{(n-1)!} t^{n-1} e^{at}}$$

2b) $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = \mathcal{U}(t-a)\, \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Big|_{t\to t-a}$

$= \mathcal{U}(t-a)\, 1\Big|_{t\to t-a}$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = \mathcal{U}(t-a)}$$

2c) $\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{(s-b)^2+k^2}\right\} = \mathcal{U}(t-a)\,\mathcal{L}^{-1}\left\{\frac{1}{(s-b)^2+k^2}\right\}_{t\to t-a}$

$= \frac{1}{k}\mathcal{U}(t-a)\left[\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\Big|_{s\to s-b}\right\}\right]_{t\to t-a}$

$= \frac{1}{k}\mathcal{U}(t-a)\left[e^{bt}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}\right]_{t\to t-a}$

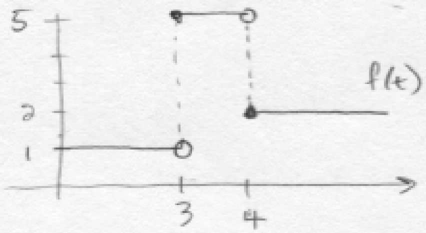
$= \frac{1}{k}\mathcal{U}(t-a)\left[e^{bt}\sin kt\right]_{t\to t-a}$

$$\boxed{\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{(s-b)^2+k^2}\right\} = \frac{1}{k}\mathcal{U}(t-a)\, e^{b(t-a)}\sin k(t-a)}$$

2d) $\boxed{\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}}$

3) Use defn of $\mathcal{L}$ to show $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

I did this in lecture. Refer to your notes

4a) Reexpress  $f(t)$ in terms of Heaviside funcs

$f(t) = 1$ since $f(t) = 1$ ~~by~~ until $t = 3$

$+4\,\mathcal{U}(t-3)$ since $f(t)$ jumps up 4 at $t = 3$

$-3\,\mathcal{U}(t-4)$ since $f(t)$ jumps down 3 at $t = 4$

$$\boxed{f(t) = 1 + 4\,\mathcal{U}(t-3) - 3\,\mathcal{U}(t-4)}$$

4b)

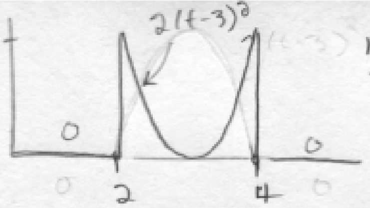
$$\mathcal{L}\{f(t)\} = \frac{1}{s} + 4\,\mathcal{L}\{\mathcal{U}(t-3)\cdot 1\} - 3\,\mathcal{L}\{\mathcal{U}(t-4)\cdot 1\}$$

$$= \frac{1}{s} + 4e^{-3t}\,\mathcal{L}\{1\} - 3e^{-4t}\,\mathcal{L}\{1\}$$

$$= \frac{1}{s} + 4e^{-3t}\,\frac{1}{s} - 3e^{-4t}\,\frac{1}{s}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{s}\left(1 + 4e^{-3t} - 3e^{-4t}\right)}$$

4b) Ditto for $f(t) =$



note that the drawing in the HW sheet is wrong!

To express this $f(t)$ in terms of Heaviside funcs, simply build a combination of Heavisides that is $\begin{cases} 1 & \text{for } 2 \le t < 4 \\ 0 & \text{elsewhere} \end{cases}$ then multiply it by $2(t-3)^2$.

$$\boxed{f(t) = 2(t-3)^2\left[\mathcal{U}(t-2) - \mathcal{U}(t-4)\right]}$$

$$f(t) = 2\,\mathcal{U}(t-2)\,(t-3)^2 - 2\,\mathcal{U}(t-4)\,(t-3)^2$$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{\mathcal{U}(t-2)((t-2)-1)^2\} - 2\mathcal{L}\{\mathcal{U}(t-4)((t-4)+1)^2\}$$

$$\mathcal{L}\{f(t)\} = 2e^{-2s}\mathcal{L}\{(t-1)^2\} - 2e^{-4s}\mathcal{L}\{(t+1)^2\}$$

$$= 2e^{-2s}\mathcal{L}\{t^2 - 2t + 1\} - 2e^{-4s}\mathcal{L}\{t^2 + 2t + 1\}$$

$$\boxed{\mathcal{L}\{f(t)\} = 2e^{-2s}\left[\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right] - 2e^{-4s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right]}$$

5) $\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

Let $F(s) = \mathcal{L}\{f(t)\}$ and $\mathcal{L}^{-1}\{F(s)\} = f(t)$

Then

$$\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-as}F(s)$$

$$\mathcal{U}(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

$$\mathcal{U}(t-a)\left[f(t)\right]_{t \to t-a} = \text{''}$$

$$\boxed{\mathcal{U}(t-a)\left[\mathcal{L}^{-1}\{F(s)\}\right]_{t \to t-a} = \mathcal{L}^{-1}\{e^{-as}F(s)\}}$$

QED.