HW #8
Math 527
Solution Set

1.) $\mathcal{L}^{-1}\left\{\frac{3}{s}+\frac{4}{s^5}-\frac{1}{s+7}\right\}=\mathcal{L}^{-1}\left\{\frac{3}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{4}{s^5}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+7}\right\}$

$=\boxed{3+\frac{1}{6}t^4-e^{-7t}}$

2.) $\mathcal{L}^{-1}\left\{\frac{5}{(s+2)(s^2+4)}\right\}=\mathcal{L}^{-1}\left\{\frac{A}{s+2}+\frac{Bs+C}{s^2+4}\right\}$

with $5=As^2+4A+Bs^2+2C+2Bs+Cs \Rightarrow \left.\begin{array}{l}4A+2C=0\\A+B=0\\2B+C=1\end{array}\right\}\Rightarrow A=-1/4,\ B=1/4,\ C=1/2$

So

$\mathcal{L}^{-1}\left\{\frac{5}{(s+2)(s^2+4)}\right\}=-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$=\boxed{\frac{1}{4}\left(-e^{-2t}+\cos 2t+\sin 2t\right)}$

3.) $\mathcal{L}^{-1}\left\{\frac{5}{s^2+2s+5}\right\}=\mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+4}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\Big|_{s\to s+1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\Big|_{s\to s+1}\right\}$

$=\boxed{e^{-t}\left(\cos 2t-\frac{1}{2}\sin 2t\right)}$

4.) $\mathcal{L}^{-1}\left\{\frac{2s+5}{(s+2)(s^2+4)}\right\}=\mathcal{L}^{-1}\left\{\frac{A}{s+2}+\frac{Bs+C}{s^2+4}\right\}$

with $2s+5=As^2+4A+Bs^2+2C+2Bs+Cs \Rightarrow \left.\begin{array}{l}A+B=0\\4A+2C=5\\2B+C=2\end{array}\right\}\Rightarrow \begin{array}{l}A=1/8\\B=-1/8\\C=9/4\end{array}$

So

$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s+2)(s^2+4)}\right\}=\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}-\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{9}{8}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$=\boxed{\frac{1}{8}\left(e^{-2t}-\cos 2t+9\sin 2t\right)}$

5.) $\mathcal{L}^{-1}\left\{\frac{1}{s+3}e^{-5s}\right\}=\mathcal{U}(t-5)f(t-5)$ with $f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$

$=e^{-3t}$

$=\boxed{e^{-3(t-5)}\mathcal{U}(t-5)}$

6.) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+5}e^{-3s}\right\} = \mathcal{U}(t-3)f(t-3)$ with $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\}$

$f(t) = \frac{1}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\}$

$f(t) = \frac{1}{\sqrt{5}}\sin\sqrt{5}t$

$= \sin(\sqrt{5}t - 3\sqrt{5})\mathcal{U}(t-3)$

$= \boxed{\frac{1}{\sqrt{5}}\sin(\sqrt{5}t - 3\sqrt{5})\mathcal{U}(t-3)}$

7.) $f(t) = 2 - 4\mathcal{U}(t-3)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} - \mathcal{L}\{4\mathcal{U}(t-3)\}$

$= \boxed{\frac{2}{s} - \frac{4}{s}e^{-3s}}$

8.) $f(t) = \sin(t)\mathcal{U}(t - \frac{3\pi}{2})$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\mathcal{U}(t-\frac{3\pi}{2})\} = e^{-\frac{3\pi}{2}s}\mathcal{L}\{\sin(t+\frac{3\pi}{2})\}$

$= e^{-\frac{3\pi}{2}s}\mathcal{L}\{-\cos(t)\} = \boxed{\frac{-s}{s^2+1}e^{-\frac{3\pi}{2}s}}$

10.) $\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t^3\}$, $y(0) = 1$, $y'(0) = 0$

$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = \frac{6}{s^4}$

$Y(s)(s^2 - 4s + 4) - s + 4 = \frac{6}{s^4} \Rightarrow Y(s) = \frac{s^5 - 4s^4 + 6}{s^4(s^2-4s+4)} = \frac{s^5 - 4s^4 + 6}{s^4(s-2)^2}$

$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s-2} + \frac{F}{(s-2)^2}$

$s^5 - 4s^4 + 6 = As^3(s-2)^2 + Bs^2(s-2)^2 + Cs(s-2)^2 + D(s-2)^2 + Es^4(s-2) + Fs^4$

$= A\check{s}^5 - 4A\check{s}^4 + 4A\check{s}^3 + B\check{s}^4 - 4B\check{s}^3 + 4B\check{s}^2 + C\check{s}^3 - 4C\check{s}^2 + 4C\check{s} + D\check{s}^2 - 4D\check{s} + 4D$
$+ E\check{s}^5 - 2E\check{s}^4 + F\check{s}^4$

$A + E = 1 \Rightarrow \boxed{E = \frac{1}{4}}$

$(-4A + B - 2E + F) = -4$
$\rightarrow +\frac{9}{8} - \frac{1}{2} + F = -4 \Rightarrow \boxed{F = -\frac{13}{8}}$

$4A - 4B + C = 0 \Rightarrow \boxed{A = \frac{3}{4}}$

$4B - 4C + D = 0 \Rightarrow \boxed{B = \frac{9}{8}}$

$4C - 4D = 0 \Rightarrow \boxed{C = \frac{3}{2}}$

$4D = 6 \Rightarrow \boxed{D = \frac{3}{2}}$

10.) continued. $Y(s) = \frac{3}{4}\frac{1}{s} + \frac{9}{8}\frac{1}{s^2} + \frac{3}{2}\frac{1}{s^3} + \frac{3}{2}\frac{1}{s^4} + \frac{1}{4}\frac{1}{s-2} - \frac{13}{8}\frac{1}{(s-2)^2}$

$y(t) = \mathcal{L}^{-1}\left\{ \frac{3}{4}\frac{1}{s} + \frac{9}{8}\frac{1}{s^2} + \frac{3}{2}\frac{1}{s^3} + \frac{2}{2}\frac{1}{s^4} + \frac{1}{4}\frac{1}{s-2} - \frac{13}{8}\frac{1}{(s-2)^2} \right\}$

$= \boxed{\frac{3}{4} + \frac{9}{8}t + \frac{3}{4}t^2 + \frac{1}{4}t^3 + \frac{1}{4}e^{2t} - \frac{13}{8}te^{2t}}$

11.) $\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t^3 e^{2t}\}$ $\quad y(0) = 0, \; y'(0) = 0$

$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = \frac{3!}{s^4}\Big|_{s \to s-2} = \frac{6}{(s-2)^4}$

$Y(s)(s^2 - 4s + 4) = \frac{6}{(s-2)^4}$

$Y(s) = \frac{6}{(s-2)^6}$ $\qquad y(t) = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{s^6}\Big|_{s \to s-2}\right\}$

$= \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\Big|_{s \to s-2}\right\}$

$= \boxed{\frac{1}{20}e^{2t}t^5}$

12.) $\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{1 - u(t-1)\}$ $\quad y(0) = 0, \; y'(0) = 1$

$s^2Y(s) + 1 + 4Y(s) = \frac{1}{s} - e^{-s}$

$Y(s)(s^2+4) = \frac{1}{s} - 1 - e^{-s}$

$Y(s) = \frac{1}{s(s^2+4)} - \frac{1}{s^2+4} - \frac{e^{-s}}{s^2+4}$

$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$A = \frac{1}{4}, \; B = -\frac{1}{4}, \; C = 0$

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{s^2+4} - \frac{e^{-s}}{s^2+4}\right\}$

$= \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t - u(t-1)\,g(t-1)$ $\quad$ where $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$= \frac{1}{2}\sin(2t)$

$= \boxed{\frac{1}{4}\left(1 - \cos 2t - 2\sin 2t - 2u(t-1)\sin 2(t-1)\right)}$