Homework 6
Solution Key
Math 527

Problem 1: $y'' + \omega^2 y = 0$ ansatz $y = e^{\lambda t}$

$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm\sqrt{-\omega^2} = \pm i\omega$

$y(x) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$

Now use Euler's Formula $e^{i\omega t} = \cos\omega t + i\sin\omega t$

$y(x) = c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t)$

$y(x) = \underbrace{(c_1+c_2)}_{c_3}\cos\omega t + \underbrace{i(c_1-c_2)}_{c_4}\sin\omega t$

$y(x) = c_3\cos\omega t + c_4\sin\omega t$

Problem 2:

$y'' + 2\mu y' + \omega_0^2 y = 0$ ansatz $y = e^{\lambda t}$

$\lambda^2 + 2\mu\lambda + \omega_0^2 = 0$

Use quadratic formula

$$\lambda = \frac{-2\mu \pm \sqrt{4\mu^2 - 4\omega_0^2}}{2} = -\mu \pm \sqrt{\mu^2 - \omega_0^2}$$

We are assuming $\mu^2 - \omega_0^2 < 0$, so $\omega_0^2 - \mu^2 = \omega^2 > 0$.

Thus $\lambda = -\mu \pm \sqrt{(-1)\omega^2} = -\mu \pm i\omega$

$y(t) = c_1 e^{(-\mu + i\omega)t} + c_2 e^{(-\mu - i\omega)t} = c_1 e^{-\mu t} e^{i\omega t} + c_2 e^{-\mu t} e^{-i\omega t}$

$y(t) = e^{-\mu t}(c_1 e^{i\omega t} + c_2 e^{-i\omega t})$

Now use Euler's Formula

$y(t) = e^{-\mu t}(c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t))$

$= e^{-\mu t}(\underbrace{(c_1+c_2)}_{c_3}\cos\omega t + \underbrace{i(c_1-c_2)}_{c_4}\sin\omega t)$

$= e^{-\mu t}(c_3\cos\omega t + c_4\sin\omega t)$

Problem 3: $my'' + ky = F_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

with initial conditions $y(0) = y'(0) = 0$.

First, let's divide by m

$$y'' + \frac{k}{m} y = \frac{F_0}{m} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

now let $\omega = \sqrt{\frac{k}{m}}$

$$y'' + \omega^2 y = \frac{F_0}{m} \sin(\omega t)$$

Solve homogeneous problem

$$\lambda^2 = -\omega^2 \Rightarrow \lambda = \pm i\omega$$

$$y_h(t) = C_1 \cos\omega t + C_2 \sin\omega t$$

Now use judicious guessing to find particular solution

$$y_p = At\sin\omega t + Bt\cos\omega t$$

$$y_p' = A\sin\omega t + A\omega t\cos\omega t + B\cos\omega t - B\omega t\sin\omega t$$

$$y_p'' = 2A\omega\cos\omega t - 2B\omega\sin\omega t - A\omega^2 t\sin\omega t - B\omega^2 t\cos\omega t$$

plug these into our equation and group by $\sin\omega t$ and $\cos\omega t$ and powers of t

$$2A\omega\cos\omega t - 2B\omega\sin\omega t + \underbrace{(\omega^2 B - \omega^2 B)}_{0} t\cos\omega t + \underbrace{(\omega^2 A - \omega^2 A)}_{0} t\sin\omega t = \frac{F_0}{m}\sin\omega t$$

$$2A\omega\cos\omega t - 2B\omega\sin\omega t = \frac{F_0}{m}\sin\omega t$$

Thus $A = 0$ and $B = \frac{-F_0}{2m\omega}$

So $y_p = \frac{-F_0}{2m\omega} t\cos\omega t$

$$\boxed{y = y_h + y_p = C_1\cos\omega t + C_2\sin\omega t - \frac{F_0}{2m\omega} t\cos\omega t}$$ general solution

Problem 3 (continued): Apply initial conditions

$0 = c_1$ from the first condition

$$y' = \omega c_2 \cos\omega t - \frac{F_0}{2m\omega}\cos\omega t + \frac{F_0}{2m} t \sin\omega t$$

$$0 = \omega c_2 - \frac{F_0}{2m\omega} \Rightarrow c_2 = \frac{F_0}{2m\omega^2}$$

$$\boxed{y(t) = \frac{F_0}{2m\omega^2}\sin\omega t - \frac{F_0}{2m\omega} t\cos\omega t}$$

Note that as t gets really large so does $y(t)$. Resonant Forcing.

Problem 4:

$$y'' - 2y' + 5y = e^x \cos x$$

Solve homogeneous problem

$$\lambda^2 - 2\lambda + 5 = 0 \qquad \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$$y_h = e^x(c_1\cos 2x + c_2 \sin 2x)$$

Use judicious guessing

$$y_p = e^x(A\cos x + B\sin x)$$

$$y_p' = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$$y_p'' = e^x(A\cos x + B\sin x) + 2e^x(-A\sin x + B\cos x) - e^x(A\cos x + B\sin x)$$

$$y_p'' = 2e^x(-A\sin x + B\cos x)$$ plug these into equation

$$-2Ae^x\sin x + 2Be^x\cos x - 2Ae^x\cos x - 2Be^x\sin x + 2Ae^x\sin x - 2Be^x\cos x + 5Ae^x\cos x + 5Be^x\sin x = e^x\cos x$$

Therefore, after cancelling some terms

$$3Be^x\sin x + 3Ae^x\cos x = e^x\cos x \Rightarrow B = 0, \quad A = \frac{1}{3}$$

$$y_p = \frac{1}{3}e^x\cos x$$

$$\boxed{y(x) = e^x(c_1\cos 2x + c_2\sin 2x) + \frac{1}{3}e^x\cos x}$$

Problem 5: $y'' - 4y = (x^2-3)\sin 2x$ $\qquad -3\tfrac{1}{4} = -\tfrac{13}{4}$

Find homogeneous solution $\quad \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$

$y_h = C_1 e^{2x} + C_2 e^{-2x}$

Guess

$$y_p = (Ax^2+Bx+C)\cos 2x + (Dx^2+Ex+F)\sin 2x$$

$$y_p' = (2Ax+B)\cos 2x - 2(Ax^2+Bx+C)\sin 2x + (2Dx+E)\sin 2x + 2(Dx^2+Ex+F)\cos 2x$$

$$y_p'' = -4(Ax^2+Bx+C)\cos 2x - 4(Dx^2+Ex+F)\sin 2x + 2A\cos 2x + 2D\sin 2x$$
$$-4(2Ax+B)\sin 2x + 4(2Dx+E)\cos 2x$$

plug in

$$-8(Ax^2+Bx+C)\cos 2x - 8(Dx^2+Ex+F)\sin 2x + 2A\cos 2x + 2D\sin 2x$$
$$-4(2Ax+B)\sin 2x + 4(2Dx+E)\cos 2x = (x^2-3)\sin 2x$$

Therefore

$-8Ax^2 - 8Bx - 8C + 2A + 8Dx + 4E = 0 \quad \leftarrow$ cos 2x terms

$-8Dx^2 - 8Ex - 8F + 2D - 8Ax - 4B = x^2 - 3 \quad \leftarrow$ sin 2x terms

And

$-8Ax^2 = 0 \Rightarrow \boxed{A=0}$ $\qquad -8Bx + 8Dx = 0 \Rightarrow \boxed{B=D}$

$2A + 4E - 8C = 0 \Rightarrow \boxed{E=2C}$

from cosine terms eqn.

$-8Dx^2 = x^2 \Rightarrow \boxed{D = -\frac{1}{8}}$

$-8Ex - 8Ax = 0 \Rightarrow \boxed{E=0}$

$-8F + 2D - 4B = -3$

$-8F - \frac{1}{4} + \frac{1}{2} = -3$

$-8F = -\frac{13}{4}$

Thus $\quad y_p = -\frac{1}{8}x\cos 2x + \frac{13}{32}\sin 2x - \frac{1}{8}x^2\sin 2x \qquad \boxed{F = \frac{13}{32}}$

$$\boxed{y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{8}x\cos 2x + \frac{13}{32}\sin 2x - \frac{1}{8}x^2\sin 2x}$$

6.) $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$

Solve for y_h

$$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

$$(\lambda+2)(\lambda-2)(\lambda-2) = 0 \qquad y_h = c_1e^{-2x} + c_2e^{2x} + c_3xe^{2x}$$

Judicious Guessing

We might try $y_p = (Ax^2 + Bx + C)e^{2x}$ but Bxe^{2x} and Ce^{2x} are both homogeneous solutions so we need to multiply by x twice

$$y_p = (Ax^4 + Bx^3 + Cx^2)e^{2x}$$

$$y_p' = 2(Ax^4 + Bx^3 + Cx^2)e^{2x} + (4Ax^3 + 3Bx^2 + 2Cx)e^{2x}$$

$$y_p'' = 4(Ax^4 + Bx^3 + Cx^2)e^{2x} + 4(4Ax^3 + 3Bx^2 + 2Cx)e^{2x} + (12Ax^2 + 6Bx + 2C)e^{2x}$$

$$y_p''' = 8(Ax^4 + Bx^3 + Cx^2)e^{2x} + 12(4Ax^3 + 3Bx^2 + 2Cx)e^{2x} + 6(12Ax^2 + 6Bx + 2C)$$
$$+ (24Ax + 6B)e^{2x}$$

plug these in to find that

$8Ax^4 - 8Ax^4 - 8Ax^4 + 8Ax^4 = 0$ ✓ x^4 terms

$(8B + 48A)x^3 + (-8B - 32A)x^3 + (-8B - 16A)x^3 + 8Bx^3 = 0$ ✓ x^3 terms

$(8C + 36B + 72A)x^2 - 2(4C + 12B + 12A)x^2 - 4(2C + 3B)x^2 + 8C = 0$ x^2 terms

$$\Rightarrow 48A = 0 \Rightarrow \boxed{A = 0} \checkmark$$

$6B + 12C - 4C = 0$ constant terms

$$8C = -6B \qquad C = -\frac{3}{4}B$$

$(24C + 36B)x - 2(8C + 6B)x - 4(2C)x = 6x$ x terms

$-18B + 36B + 12B - 12B + 6B = 6$

$\cancel{-12B + 36B + 3B - 12B + 4B = 6}$

$24B = 6$

$$\boxed{B = \frac{1}{4}} \qquad \boxed{C = -\frac{3}{16}}$$

So $y_p = \left(\frac{1}{4}x^3 - \frac{3}{16}x^2\right)e^{2x}$

$$\boxed{y = y_h + y_p = c_1e^{-2x} + c_2e^{2x} + c_3xe^{2x} + \left(\frac{1}{4}x^3 - \frac{3}{16}x^2\right)e^{2x}}$$

Problem 7: $y'' + 2y' + y = e^{-t}\ln(t)$

Solve homogeneous

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda = -1 \quad \text{repeated root} \qquad y_h = c_1 e^{-t} + c_2 t e^{-t}$$

$y_1 = e^{-t}$ $\qquad y_2 = te^{-t}$ $\qquad f = e^{-t}\ln(t)$

Use variation of parameters

$$W(y_1, y_2) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = e^{-2t}(1 - t + t) = e^{-2t}$$

$$u_1' = \frac{-y_2 f}{W(y_1,y_2)} = -e^{2t} t e^{-t} e^{-t} \ln(t) = -t\ln(t)$$

$$u_1 = \int t\ln(t)\,dt = \frac{-1}{4}t^2(2\ln(t) - 1)$$

$$u_2' = \frac{y_1 f}{W(y_1,y_2)} = e^{2t} e^{-t} e^{-t}\ln(t) = \ln(t)$$

$$u_2 = \int \ln(t)\,dt = t(\ln(t) - 1)$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{-1}{4}t^2 e^{-t}(2\ln(t) - 1) + t^2 e^{-t}(\ln(t) - 1)$$

$$y_p = t^2 e^{-t}\left(-\frac{1}{2}\ln(t) + \frac{1}{4} + \ln(t) - 1\right)$$

$$= t^2 e^{-t}\left(\frac{1}{2}\ln(t) - \frac{3}{4}\right)$$

$$y_p = \frac{1}{4}t^2 e^{-t}(2\ln(t) - 3)$$

$$\boxed{y = y_h + y_p = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{4}t^2 e^{-t}(2\ln(t) - 3)}$$

Problem 8: $y'' + y = \cos^2 t$

Solve homogeneous problem

$\lambda^2 + 1 = 0 \qquad \lambda = \pm i \qquad y_h = C_1 \cos t + C_2 \sin t$

Use variation of parameters

$y_1 = \cos t \qquad y_2 = \sin t \qquad f = \cos^2 t$

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$u_1' = -y_2 f = -\sin t \cos^2 t$ use u-substitution $u = \cos t$, $du = -\sin t\, dt$

$u_1 = \int u_1'\, dt = -\int \sin t \cos^2 t\, dt = \frac{1}{3}\cos^3 t$

$u_2' = \cancel{}\ y_1 f = \cos^3 t$ int by parts $u = \cos^2 x$, $dv = \cos x$ u-sub $u = \sin x$

$u_2 = \int u_2'\, dt = \int \cos^3 t\, dt = \sin t \cos^2 t + 2\int \cos t \sin^2 t\, dt$

$u_2 = \sin x \cos^2 t + \frac{2}{3}\sin^3 t$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{3}\cos^4 t + \sin^2 t \cos^2 t + \frac{2}{3}\sin^4 t$$

$$= \left(\frac{2}{3}\sin^2 t + \frac{1}{3}\cos^2 t\right)\underbrace{(\sin^2 t + \cos^2 t)}_{1}$$

$$= \frac{1}{3}\underbrace{(\sin^2 t + \cos^2 t)}_{1} + \frac{1}{3}\sin^2 t$$

$$= \frac{1}{3}(1 + \sin^2 t)$$

$y = y_h + y_p$

$y = C_1 \cos t + C_2 \sin t + \frac{1}{3}(1 + \sin^2 t)$

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Problem 9:

$$3y'' - 6y' + 6y = e^x \sec x$$

· put in standard form

$$y'' - 2y' + 2y = \frac{1}{3} e^x \sec x$$

$$\lambda^2 - 2\lambda + 2 = 0 \qquad \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$y_h = e^x c_1 \cos x + e^x c_2 \sin x$$

Use variation of parameters

$$y_1 = e^x \cos x \qquad y_2 = e^x \sin x \qquad f = \frac{1}{3} e^x \sec x$$

$$W(y_1, y_2) = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix} = e^{2x}\left(\cos x \sin x + \cos^2 x - \sin x \cos x + \sin^2 x\right)$$

$$= e^{2x}$$

$$u_1' = \frac{1}{e^{2x}}\left(-y_2 f\right) = \frac{-1}{3} e^{-2x} e^x \sin x \, e^x \sec x$$

$$= -\frac{1}{3} \frac{\sin x}{\cos x}$$

$$u_1 = -\int \frac{1}{3} \frac{\sin x}{\cos x} = -\frac{1}{3} \ln(\cos x) \qquad \text{by u-sub } u = \cos x$$

$$u_2' = \frac{1}{3} e^{-2x} e^x \cos x \, e^x \sec x$$

$$= \frac{1}{3} \frac{\cos x}{\cos x} = \frac{1}{3}$$

$$u_2 = \frac{1}{3} \int 1 \, dx = \frac{1}{3} x$$

$$y_p = \frac{1}{3} x e^x \sin x - \frac{1}{3} e^x \cos x \ln(\cos x) = \frac{1}{3} e^x \left(x \sin x - \cos x \ln(\cos x)\right)$$

$$y = e^x \left(c_1 \cos x + c_2 \sin x + \frac{1}{3}\left(x \sin x - \cos x \ln(\cos x)\right)\right)$$