Homework # 2 Solution Set Math 527

Find the general solution of these odes.

1.) $\frac{dy}{dt} = e^{t+y+3}$ use separation of variables

$\frac{dy}{dt} = e^{t+3} e^{y}$

$e^{-y} \frac{dy}{dt} = e^{t+3}$

$\int e^{-y} \frac{dy}{dt}\, dt = \int \frac{d}{dt}\left[-e^{-y}\right] dt = \int e^{t+3}\, dt$ $\int e^{-y}\, dy = -e^{-y}$

$-e^{-y} + C_1 = e^{t+3} + C_2$

$e^{-y} = -e^{t+3} + C$

$\boxed{y = -\ln(-e^{t+3} + C)}$ Solution is valid for $t \leq \ln(C) - 3$

2.) $\frac{dy}{dx} = xy + 3x + y + 3$

$\frac{dy}{dx} = (x+1)(y+3)$

$(y+3)^{-1} \frac{dy}{dx} = (x+1)$

$\int (y+3)^{-1} \frac{dy}{dx}\, dx = \int (x+1)\, dx$ $\int (y+3)^{-1}\, dy = \ln|y+3|$

$\int \frac{d}{dx}\left[\ln|y+3|\right] dx = \int (x+1)\, dx$

$\ln|y+3| = \frac{1}{2}x^2 + x + C_1$

$|y+3| = e^{\frac{1}{2}x^2 + x + C_1} = C_2 e^{\frac{1}{2}x^2 + x}$

The absolute value gives us two cases:

Case 1: $(y+3) > 0$ Case 2: $(y+3) < 0$

then $y + 3 = C_2 e^{\frac{1}{2}x^2 + x}$ then $-(y+3) = C_2 e^{\frac{1}{2}x^2 + x}$

$y = C_2 e^{1/2 x^2 + x} - 3$ $y = -C_2 e^{\frac{1}{2}x^2 + x} - 3$

But we have an arbitrary constant, so both cases can be pulled into one:

$\boxed{y = Ce^{\frac{1}{2}x^2 + x} - 3}$

3.) $\cos y \sin t \frac{dy}{dt} = \sin y \cos t$ use separation of variables

$$\frac{\cos y}{\sin y}\frac{dy}{dt} = \frac{\cos t}{\sin t}$$

$$\int \frac{\cos y}{\sin y}\frac{dy}{dt}\,dt = \int \frac{\cos t}{\sin t}\,dt$$

$$\int \frac{d}{dt}\left[\int \frac{\cos y}{\sin y}\,dy\right]dt = \int \frac{\cos t}{\sin t}\,dt$$

$$\int \frac{\cos y}{\sin y}\,dy + C_1 = \int \frac{\cos t}{\sin t}\,dt$$

use u-substitution for both integrals, $u = \sin y$ or $\sin t$ respectively

$$\ln|\sin y| + C_2 = \ln|\sin t| + C_3$$

$$\ln|\sin y| = \ln|\sin t| + C_4$$

$$|\sin y| = e^{\ln|\sin t| + C_4} = C_5 e^{\ln|\sin t|} = C_5|\sin t|$$

$$|\sin y| = C_5|\sin t|$$

The absolute values give us $\sin y = \pm C_5 \sin t$ but we can pull the $\pm$ into the arbitrary constant.

$$\sin y = c\sin t$$

$$\boxed{y = \sin^{-1}(c\sin t)}$$

4.) $\frac{dy}{dx} = \frac{x - e^{x}}{y + e^{y}}$ Separation of Variables

$$(y + e^{y})\frac{dy}{dx} = x - e^{-x}$$

$$\int (y + e^{y})\,dy = \int (x - e^{-x})\,dx$$

$$\frac{1}{2}y^2 + e^{y} + C_1 = \frac{1}{2}x^2 + e^{-x} + C_2$$

$$\boxed{\frac{1}{2}y^2 + e^{y} = \frac{1}{2}x^2 + e^{-x} + C}$$

5.) $\frac{dy}{dx} + y = xe^x$ Use integrating factor

Already in standard form, so $P(x) = 1$. The integrating factor

$u(x) = e^{\int 1\,dx} = e^x$

$e^x\left(\frac{dy}{dx} + y\right) = e^x xe^x$

$\int e^x\left(\frac{dy}{dx} + y\right)dx = \int xe^{2x}\,dx$

Int by parts

$\int \frac{d}{dx}\left[e^x y\right]dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C_1$

$e^x y = \frac{1}{2}e^{2x}\left(x - \frac{1}{2}\right) + C_2$

$\boxed{y = \frac{1}{2}e^x\left(x - \frac{1}{2}\right) + Ce^{-x}}$

6.) $\frac{dy}{dt} + t^2 y = 1$ Integrating factor

$u(x) = e^{\int t^2\,dt} = e^{\frac{1}{3}t^3}$

$e^{\frac{1}{3}t^3}\left(\frac{dy}{dt} + t^2 y\right) = e^{\frac{1}{3}t^3}$

$\int e^{\frac{1}{3}t^3}\left(\frac{dy}{dt} + t^2 y\right)dt = \int e^{\frac{1}{3}t^3}\,dt$

$\int \frac{d}{dt}\left[e^{\frac{1}{3}t^3} y\right]dt = \int e^{\frac{1}{3}t^3}\,dt$

$e^{\frac{1}{3}t^3} y + C_1 = \int e^{1/3 t^3}\,dt$

$\boxed{y = e^{-\frac{1}{3}t^3}\int e^{\frac{1}{3}t^3}\,dt}$

Fully simplified

Note that C_1 is unnecessary because the remaining integral will have its own constant of integration.

7.) $\frac{dy}{dt} + \frac{2t}{1+t^2} y = \frac{1}{1+t^2}$ Use integrating factor

$u(x) = e^{\int \frac{2t}{1+t^2} dt}$ u-substitution, $u = 1+t^2$ for integral

$u(x) = e^{\ln(1+t^2)} = |1+t^2| = 1+t^2$

$(1+t^2)\left(\frac{dy}{dt} + \frac{2t}{1+t^2} y\right) = \frac{1}{1+t^2}(1+t^2)$

$\frac{d}{dt}\left[(1+t^2)y\right] = \frac{1+t^2}{1+t^2} = 1$

$\int \frac{d}{dt}\left[(1+t^2)y\right] dt = \int 1 \, dt$

$(1+t^2)y + C_1 = t + C_2$

$$\boxed{y = \frac{t+C}{1+t^2}}$$

8.) $x^2(1+y^2) + 2\frac{dy}{dx} = 0$, $y(0) = 1$

$\frac{dy}{dx} = -\frac{1}{2}x^2(1+y^2)$

$\frac{1}{1+y^2}\frac{dy}{dx} = -\frac{1}{2}x^2$

$\int \frac{dy}{1+y^2} = \int -\frac{1}{2}x^2 \, dx$

$\tan^{-1} y + C_1 = -\frac{1}{6}x^3 + C_2$

$y = \tan\left(-\frac{1}{6}x^3 + C\right)$

Now we'll use our initial value to solve for C

$1 = \tan\left(-\frac{1}{6}0^3 + C\right) = \tan(C)$

Thus $C = \frac{(n+1)\pi}{4}$ $n = \ldots -2, -1, 0, 1, 2, \ldots$

Any choice for n works, $n=0$ is nice.

$$\boxed{y = \tan\left(\frac{\pi}{4} - \frac{1}{6}x^3\right)}$$

8.) $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$, $y(0)=-1$

$2(y-1)\frac{dy}{dx} = 3x^2+4x+2$

$\int 2(y-1)\,dy = \int 3x^2+4x+2\,dx$

$2(y^2/2 - y) + C_1 = x^3+2x^2+2x+C_2$

$y^2-2y = x^3+2x^2+2x+C_3$

$(y-1)^2-1 = x^3+2x^2+2x+C_3$

$(y-1)^2 = x^3+2x^2+2x+C_4$

$y-1 = \pm\sqrt{x^3+2x^2+2x+C_4}$

$y = 1\pm\sqrt{x^3+2x^2+2x+C}$

Now use the initial value

$-1 = 1\pm\sqrt{C}$

$-2 = \pm\sqrt{C}$ The square root is always positive so we want to choose the minus, and $C=4$.

$-2 = -\sqrt{4}$

$\boxed{y = 1-\sqrt{x^3+2x^2+2x+4}}$

9.) $\frac{dy}{dt} = ky$, $y(0)=y_0$

$\frac{1}{y}\frac{dy}{dt} = k$

$\frac{d}{dt}\int\frac{1}{y}\,dy = k$

$\int\frac{d}{dt}\ln|y|\,dt = \int k\,dt$

$\ln|y| = kt+C_1$

$|y| = e^{kt+C_1}$

$y = Ce^{kt}$

Now use initial value

$y_0 = Ce^0 = C$

So

$\boxed{y = y_0e^{kt}}$