Problem 1:

Homework 12 Solutions
Math 527

Suppose that the n^{th}-order linear system $\underline{x}' = A\underline{x}$ has a pair of complex eigenvalues $\lambda_1 = \mu + i\omega$ and $\lambda_2 = \mu - i\omega$ and corresponding eigenvectors $\underline{v}_1 = \underline{a} + i\underline{b}$ and $\underline{v}_2 = \underline{a} - i\underline{b}$.

We want to rewrite the solution

$$\underline{x}(t) = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t}$$

in terms of sines and cosines. Then

$$\underline{x}(t) = C_1(\underline{a} + i\underline{b}) e^{(\mu + i\omega)t} + C_2(\underline{a} - i\underline{b}) e^{(\mu - i\omega)t}$$

$$\underline{x}(t) = C_1(\underline{a} + i\underline{b}) e^{\mu t} e^{i\omega t} + C_2(\underline{a} - i\underline{b}) e^{\mu t} e^{-i\omega t}$$

And using Euler's formula: $e^{i\omega t} = \cos\omega t + i\sin\omega t$

$$\underline{x}(t) = C_1(\underline{a} + i\underline{b}) e^{\mu t}(\cos\omega t + i\sin\omega t) + C_2(\underline{a} - i\underline{b}) e^{\mu t}(\cos\omega t - i\sin\omega t)$$

Multiplying and separating real and imaginary parts

$$\underline{x}(t) = \left(C_1 \underline{a}\cos\omega t + C_2 \underline{a}\cos\omega t - C_1 \underline{b}\sin\omega t - C_2 \underline{b}\sin\omega t\right) e^{\mu t}$$

$$+ i\left(C_1 \underline{a}\sin\omega t - C_2 \underline{a}\sin\omega t + C_1 \underline{b}\cos\omega t - C_2 \underline{b}\cos\omega t\right) e^{\mu t}$$

$$\underline{x}(t) = (C_1 + C_2)(\underline{a}\cos\omega t - \underline{b}\sin\omega t) e^{\mu t} + i(C_1 - C_2)(\underline{a}\sin\omega t + \underline{b}\cos\omega t) e^{\mu t}$$

Now C_1, C_2 and i are all just constants so we can call

$C_1 + C_2 = c_1$ and $i(C_1 - C_2) = c_2$ (note that little c_1 is not equal to big C_1 etc.)

Thus

$$\underline{x}(t) = c_1(\underline{a}\cos\omega t - \underline{b}\sin\omega t) e^{\mu t} + c_2(\underline{a}\sin\omega t + \underline{b}\cos\omega t) e^{\mu t}$$

$\square$

Problem 2: $\quad \underline{x}' = \underbrace{\begin{pmatrix} -10 & -5 \\ 8 & 12 \end{pmatrix}}_{A} \underline{x}$

We make the ansatz that $\underline{x}(t) = \underline{v} e^{\lambda t}$, thus

$$A \underline{v} e^{\lambda t} = \lambda \underline{v} e^{\lambda t} \Rightarrow A \underline{v} = \lambda \underline{v}$$ which is the eigenvalue equation for A.

Find the eigenvalues of A: (use $\det(A - \lambda I) = 0$)

$$\det(A - \lambda I) = \det\begin{pmatrix} -10-\lambda & -5 \\ 8 & 12-\lambda \end{pmatrix} = \lambda^2 - 2\lambda - 120 + 40 = 0$$

$$\lambda^2 - 2\lambda - 80 = 0 \qquad \text{quadratic formula: } \lambda = \frac{2 \pm \sqrt{4 - 320}}{2} = 1 \pm 9$$

So $\boxed{\lambda_1 = 10 \qquad \lambda_2 = -8}$

Find the corresponding eigenvectors: (use $(A - \lambda I)\underline{v} = \underline{0}$)

For λ_1

$$\begin{pmatrix} -20 & -5 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ Use Gaussian Elimination (or any other means) to solve this system.

$$\left(\begin{array}{cc|c} -20 & -5 & 0 \\ 8 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} v_1 + 1/4\, v_2 = 0 \Rightarrow v_2 = -4 v_1 \\ 0 = 0 \end{array}$$

This tells us that our eigenvector has the form

$\underline{v}_1 = \begin{pmatrix} v_1 \\ -4v_1 \end{pmatrix}$ where v_1 can be anything. But we only need one specific instance of $\underline{v}_1$, so we'll choose $v_1 = 1 \Rightarrow \boxed{\underline{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}}$

For λ_2

$$\begin{pmatrix} -2 & -5 \\ 8 & 20 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & -5 & 0 \\ 8 & 20 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 5/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_1 = -5/2\, v_2$$

So $\boxed{\underline{v}_2 = \begin{pmatrix} -5/2 \\ 1 \end{pmatrix}}$ is a good choice for $\underline{v}_2$.

Thus our general solution is

$$\boxed{\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{10t} + c_2 \begin{pmatrix} -5/2 \\ 1 \end{pmatrix} e^{-8t}}$$

Problem 3: $\underline{x}' = \begin{pmatrix} 6 & 1 \\ -5 & 4 \end{pmatrix} \underline{x}$

Find the eigenvalues:

$$\det \begin{pmatrix} 6-\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} = \lambda^2 - 10\lambda + 24 + 5 = \lambda^2 - 10\lambda + 29$$

quadratic formula: $\lambda = \dfrac{10 \pm \sqrt{100 - 116}}{2} = 5 \pm 2i$

$$\boxed{\lambda_1 = 5 + 2i \qquad \lambda_2 = 5 - 2i}$$

Find corresponding eigenvectors:

For λ_1 $\begin{pmatrix} 1-2i & 1 \\ -5 & -1-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ $(1-2i)v_1 + v_2 = 0 \Rightarrow v_2 = (-1+2i)v_1$

So $-5v_1 + (-1-2i)(-1+2i)v_1 = 0$

$-5v_1 + 5v_1 = 0$ ✓

$v_2 = (-1+2i)v_1$

Thus choosing $v_1 = 1$, $\underline{v}_1 = \begin{pmatrix} 1 \\ -1+2i \end{pmatrix}$ is a valid eigenvector.

For λ_2 $\begin{pmatrix} 1+2i & 1 \\ -5 & -1+2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $(1+2i)v_1 + v_2 = 0$

So $v_2 = (-1-2i)v_1$

The second equation becomes $-5v_1 + (-1+2i)(-1-2i)v_1 = -5v_1 + 5v_1 = 0$, so

$\underline{v}_2 = \begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$ works.

Thus the general solution in terms of complex exponentials is

$$\boxed{\underline{x}(t) = C_1 \begin{pmatrix} 1 \\ -1+2i \end{pmatrix} e^{(5+2i)t} + C_2 \begin{pmatrix} 1 \\ -1-2i \end{pmatrix} e^{(5-2i)t}}$$

For general sol'n in terms of sines and cosines use the result from problem 1, with

$\underline{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\mu = 5$ and $\omega = 2$, thus

$$\underline{x}(t) = c_1 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin 2t \right] e^{5t} + c_2 \left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin 2t \right] e^{5t}$$

or $\underline{x}(t) = c_1 \begin{pmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{pmatrix}$

4.) $\underline{x}' = \begin{pmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}$

Find eigenvalues:

$$\det\begin{pmatrix} 2-\lambda & 4 & 4 \\ -1 & -2-\lambda & 0 \\ -1 & 0 & -2-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{pmatrix} - 4\det\begin{pmatrix} -1 & 0 \\ -1 & -2-\lambda \end{pmatrix} + 4\det\begin{pmatrix} -1 & -2-\lambda \\ -1 & 0 \end{pmatrix}$$

$$= (2-\lambda)(-2-\lambda)^2 - 4(2+\lambda) + 4(-2-\lambda)$$

$$= (2-\lambda)(\lambda^2+4\lambda+4) - 8 - 4\lambda - 8 - 4\lambda$$

$$= -\lambda^3 - 4\lambda^2 - 4\lambda + 2\lambda^2 + 8\lambda + 8 - 16 - 8\lambda$$

$$= -\lambda^3 - 2\lambda^2 - 4\lambda - 8 = 0$$

$\lambda = -2$ is a solution, so $(\lambda+2)\dfrac{-\lambda^3 - 2\lambda^2 - 4\lambda - 8}{\lambda+2} = 0$

$$\begin{array}{r|l} & -\lambda^2 - 4 \\ \lambda+2 & -\lambda^3 - 2\lambda^2 - 4\lambda - 8 \\ & +\lambda^3 + 2\lambda^2 \\ \hline & 0 + 0 - 4\lambda - 8 \end{array}$$

$\Rightarrow$

$$(\lambda+2)(-\lambda^2-4) = 0$$

$$(\lambda+2)(\lambda^2+4) = 0$$

$$(\lambda+2)(\lambda+2i)(\lambda-2i) = 0$$

So the eigenvalues are: $\boxed{\lambda_1 = -2,\ \lambda_2 = 2i \text{ and } \lambda_3 = -2i}$

Find corresponding eigenvectors:

For λ_1

$$\left(\begin{array}{ccc|c} 4 & 4 & 4 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $v_1 = 0$

$v_2 = -v_3$ so well let $v_2 = 1$ and $\boxed{\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$

and $0 = 0$

$\boxed{\text{For } \lambda_2}$

$$\left(\begin{array}{ccc|c} 2-2i & 4 & 4 & 0 \\ -1 & -2-2i & 0 & 0 \\ -1 & 0 & -2-2i & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2+2i & 0 & 0 \\ 2-2i & 4 & 4 & 0 \\ -1 & 0 & -2-2i & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2+2i & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 2+2i & -2-2i & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2+2i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So $v_1 = (-2-2i)v_2$ and $v_3 = v_2$

Let $v_2 = 1$ and $\boxed{\underline{v}_2 = \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix}}$

$\boxed{\text{For } \lambda_3}$ λ_2 and λ_3 are a complex conjugate pair, thus we know that their eigenvectors are also complex conjugates. Thus

$$\boxed{\underline{v}_3 = \begin{pmatrix} -2+2i \\ 1 \\ 1 \end{pmatrix}}$$

Thus we have a general solution in terms of complex exponentials

$$\boxed{\underline{x}(t) = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix} e^{2it} + c_3 \begin{pmatrix} -2+2i \\ 1 \\ 1 \end{pmatrix} e^{-2it}}$$

This can be rewritten in terms of sines and cosines as,

$\underline{a} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$, $\mu = 0$, $\omega = 2$

$$\boxed{\underline{x}(t) = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t} + \tilde{c}_2 \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \sin 2t \right] + \tilde{c}_3 \left[\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \sin 2t \right]}$$

5.) $(1-x)y'' + y = 0$ Ansatz: $y = \sum_{n=0}^{\infty} c_n x^n$

$$(1-x)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

$$\sum_{m=0}^{\infty}(m+2)(m+1)c_{m+2}x^m - \sum_{m=0}^{\infty}(m+1)m\,c_{m+1}x^m + \sum_{m=0}^{\infty} c_m x^m = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2} - (n+1)n\,c_{n+1} + c_n\right]x^n = 0$$

Thus we have a recurrence relation

$$c_{n+2} = \frac{(n+1)n\,c_{n+1} - c_n}{(n+2)(n+1)} \qquad \text{for } n = 0, 1, 2, \ldots$$

For $n=0$: $c_2 = \dfrac{-c_0}{2}$

For $n=1$: $c_3 = \dfrac{2c_2 - c_1}{3\cdot 2} = \dfrac{-c_0 - c_1}{3\cdot 2}$

For $n=2$: $c_4 = \dfrac{3\cdot 2\,c_3 - c_2}{4\cdot 3} = \dfrac{-c_0 - c_1 + \frac{c_0}{2}}{4\cdot 3} = \dfrac{-\frac{c_0}{2} - c_1}{4\cdot 3} = \dfrac{-c_0 - 2c_1}{4\cdot 3\cdot 2}$

For $n=3$: $c_5 = \dfrac{4\cdot 3\,c_4 - c_3}{5\cdot 4} = \dfrac{-\frac{c_0}{2} - c_1 + \frac{c_0 + c_1}{3\cdot 2}}{5\cdot 4} = \dfrac{-2c_0 - 5c_1}{5\cdot 4\cdot 3\cdot 2}$

For $n=4$: $c_6 = \dfrac{5\cdot 4\,c_5 - c_4}{6\cdot 5} = \dfrac{\frac{-2c_0 - 5c_1}{3\cdot 2} + \frac{c_0 + 2c_1}{4\cdot 3\cdot 2}}{6\cdot 5} = \dfrac{-7c_0 - 18c_1}{6\cdot 5\cdot 4\cdot 3\cdot 2}$

...

Thus we can write the general solution as

$$y(x) = c_0 + c_1 x - \frac{c_0}{2}x^2 - \frac{c_0 + c_1}{3\cdot 2}x^3 - \frac{c_0 + 2c_1}{4\cdot 3\cdot 2}x^4 - \frac{2c_0 + 5c_1}{5!}x^5 - \cdots$$

$$= c_0\left[1 - \frac{x^2}{2} - \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{2x^5}{5!} - \frac{7x^6}{6!} \cdots\right] + c_1\left[x - \frac{x^3}{3!} - \frac{2x^4}{4!} - \frac{5x^5}{5!} - \frac{18x^6}{6!} \cdots\right]$$

6.) $y'' + x^2 y = 0$ Ansatz: $y = \sum_{n=0}^{\infty} c_n x^n$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

$$2c_2 + 3\cdot 2 c_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) c_{n+2} + c_{n-2} \right] x^n = 0$$

Thus $c_2 = 0$, $c_3 = 0$ and $c_{n+2} = \dfrac{-c_{n-2}}{(n+2)(n+1)}$ for $n = 2, 3, 4, \ldots$

For $n=2$: $c_4 = \dfrac{-c_0}{4\cdot 3}$

For $n=3$: $c_5 = \dfrac{-c_1}{5\cdot 4}$

For $n=4$: $c_6 = 0 = c_{10} = c_{14} = c_{18} = \cdots$

For $n=5$: $c_7 = 0 = c_{11} = c_{15} = c_{19} = \cdots$

$$c_8 = \frac{-c_4}{8\cdot 7} = \frac{c_0}{8\cdot 7\cdot 4\cdot 3}$$

$$c_9 = \frac{-c_5}{9\cdot 8} = \frac{c_1}{9\cdot 8\cdot 5\cdot 4}$$

$$c_{12} = \frac{-c_8}{12\cdot 11} = \frac{-c_0}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}$$

$$c_{13} = \frac{-c_9}{13\cdot 12} = \frac{-c_1}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}$$

$\vdots$

So

$$y(x) = c_0 + c_1 x - \frac{c_0}{4\cdot 3} x^4 - \frac{c_1}{5\cdot 4} x^5 + \frac{c_0}{8\cdot 7\cdot 4\cdot 3} x^8 + \frac{c_1}{9\cdot 8\cdot 5\cdot 4} x^9 + \cdots$$

$$= c_0 \left[1 - \frac{x^4}{4\cdot 3} + \frac{x^8}{8\cdot 7\cdot 4\cdot 3} - \frac{x^{12}}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} + \cdots \right] + c_1 \left[x - \frac{x^5}{5\cdot 4} + \frac{x^9}{9\cdot 8\cdot 5\cdot 4} - \cdots \right]$$