Math 527
HW 10 Solutions

1.) $y'' - xy = 0$

Ansatz: $y = \sum_{n=0}^{\infty} c_n x^n$ plug this into equation

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\sum_{n=-1}^{\infty} (n+3)(n+2)c_{n+3}x^{n+1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2)c_{n+3}x^{n+1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$2c_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2)c_{n+3} - c_n\right] x^{n+1} = 0$$

Because this must hold for all values of x, we know that:

$2c_2 = 0$ $\qquad (n+3)(n+2)c_{n+3} - c_n = 0$ for all $n = 0, 1, 2, \ldots$

Thus $c_2 = 0$ and $c_{n+3} = \dfrac{c_n}{(n+3)(n+2)}$ ← Recurrence Relation

$n=0$: $c_3 = \dfrac{c_0}{3\cdot 2}$

$n=1$: $c_4 = \dfrac{c_1}{4\cdot 3}$

$n=2$: $c_5 = 0 = c_8 = c_{11} = \cdots$

$n=3$: $c_6 = \dfrac{c_3}{6\cdot 5} = \dfrac{c_0}{6\cdot 5\cdot 3\cdot 2}$

$n=4$: $c_7 = \dfrac{c_4}{7\cdot 6} = \dfrac{c_1}{7\cdot 6\cdot 4\cdot 3}$

$n=6$: $c_9 = \dfrac{c_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}$

$n=7$: $c_{10} = \dfrac{c_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$

So

$$y(x) = c_0 + c_1 x + \frac{c_0}{3\cdot 2}x^3 + \frac{c_1}{4\cdot 3}x^4 + \frac{c_0}{6\cdot 5\cdot 3\cdot 2}x^6 + \frac{c_1}{7\cdot 6\cdot 4\cdot 3}x^7 + \cdots$$

regrouping

$$y(x) = c_0\left(1 + \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} + \frac{x^9}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} + \cdots\right) + c_1\left(x + \frac{x^4}{4\cdot 3} + \frac{x^7}{7\cdot 6\cdot 4\cdot 3} + \frac{x^{10}}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} + \cdots\right)$$

$$y(x) = c_0 y_1(x) + c_1 y_2(x)$$

y_1 and y_2 are two linearly independent solutions.

This equation has no singular points and thus ~~the~~ no lower bound on the radius of convergence is infinite. The solutions converge everywhere.

2.) $y'' + x^2 y' + xy = 0$ Ansatz: $y = \sum_{n=0}^{\infty} c_n x^n$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$2c_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2) c_{n+3} + (n+1) c_n \right] x^{n+1} = 0.$$

Thus $c_2 = 0$ and we have a recurrence relation

$$c_{n+3} = \frac{-(n+1)c_n}{(n+3)(n+2)}$$

$n=0$: $c_3 = \frac{-c_0}{3\cdot 2}$ $n=1$: $c_4 = \frac{-2c_1}{4\cdot 3}$ $n=2$: $c_5 = 0$

$n=3$: $c_6 = \frac{4c_0}{6\cdot 5\cdot 3\cdot 2} = \frac{4^2 c_0}{6!}$ $n=4$: $c_7 = \frac{5\cdot 2 c_1}{7\cdot 6\cdot 4\cdot 3} = \frac{5^2\cdot 2^2 c_1}{7!}$

$n=6$: $c_9 = \frac{-7^2\cdot 4^2 c_0}{9!}$ $n=7$: $c_{10} = \frac{-8^2\cdot 5^2\cdot 2^2 c_1}{10!}$ ...

So we have two linearly independent solutions

$$y_1(x) = 1 - \frac{1}{3!}x^3 + \frac{4^2}{6!}x^6 - \frac{7^2\cdot 4^2}{9!}x^9 + \cdots$$

$$y_2(x) = x - \frac{2^2}{4!}x^4 + \frac{5^2\cdot 2^2}{7!}x^7 - \frac{8^2\cdot 5^2\cdot 2^2}{10!}x^{10} + \cdots$$

With a general solution

$$y(x) = c_0 y_1(x) + c_1 y_2(x)$$

Again there are no singular points and thus the lower bound on the radius of convergence is infinite.

3.) $(x-1)y'' + y' = 0$ Ansatz: $y = \sum_{n=0}^{\infty} c_n x^n$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} \left[n(n-1)c_n - (n+1)n c_{n+1} + n c_n \right] x^{n-1} = 0$$

Recurrence Relation: $c_{n+1} = \dfrac{n + n(n-1)}{n(n+1)} c_n = \dfrac{n}{n+1} c_n$

$n=1$: $c_2 = \frac{1}{2} c_1$ $n=2$: $c_3 = \frac{1}{3} c_1$ $n=3$: $c_4 = \frac{1}{4} c_1$

$n=4$: $c_5 = \frac{1}{5} c_1$ ...

$$c_n = \frac{1}{n} c_1$$

Thus our general solution is

$$y(x) = c_0 + c_1 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_1 x^3 + \cdots$$

$$y(x) = c_0 + c_1 \sum_{n=1}^{\infty} \frac{x^n}{n}$$

And we have two linearly independent solutions

$y_1(x) = 1$ $y_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ $y(x) = c_0 y_1 + c_1 y_2$

To find a lower bound on the radius of convergence we look at the singular points. In standard form the equation is

$y'' + \frac{1}{x-1} y' = 0$ $\frac{1}{x-1}$ is not defined at $x=1$, so this is a singular point.

The distance from the point around which we are expanding our power series to the singular point is $|0-1| = 1$.

Lower bound on radius of convergence: 1.

4.) $(x^2+2)y'' - 6y = 0$ Ansatz! $y = \sum_{n=0}^{\infty} c_n x^n$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 6\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[n(n-1)c_n + 2(n+2)(n+1)c_{n+2} - 6c_n \right] x^n = 0$$

Recurrence Relation: $c_{n+2} = \dfrac{6 - n(n-1)}{2(n+2)(n+1)} c_n = \dfrac{3-n}{2(n+1)} c_n$

$c_2 = \frac{3}{2} c_0$

$c_3 = \frac{1}{2} c_1$

$c_4 = \frac{1}{4} c_0$

$c_5 = 0 = c_7 = c_9 = \cdots$

$c_6 = \dfrac{-c_0}{2 \cdot 4 \cdot 5}$

$c_8 = \dfrac{3c_0}{2 \cdot 2 \cdot 4 \cdot 5 \cdot 7}$

$c_{10} = \dfrac{-3c_0}{2 \cdot 2 \cdot 2 \cdot 4 \cdot 7 \cdot 9}$

We have two lin. indpt. solns

$$y_1(x) = 1 + \frac{3}{2}x^2 + \frac{1}{4}x^4 - \frac{x^6}{2 \cdot 4 \cdot 5} + \frac{3x^8}{2 \cdot 2 \cdot 4 \cdot 5 \cdot 7} + \cdots$$

$$y_1(x) = \sum_{n=0}^{\infty} \frac{3(-1)^n}{2^n(2n-1)(2n-3)} x^{2n}$$

$$y_2(x) = x + \frac{1}{2}x^3$$

$$y(x) = c_0 y_1 + c_1 y_2$$

There are singular points where $x^2+2=0$. This happens for $x = \pm\sqrt{2}\,i$. The distance from the origin ($x=0$, the center of our series) to $\pm\sqrt{2}\,i$ is $\sqrt{2}$. Thus the lower bound on the radius of convergence is $\sqrt{2}$.

5.) $(x-1)y'' - xy' + y = 0 \qquad y(0) = -2 \qquad y'(0) = 6$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} (n+1)n c_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$c_0 - 2c_2 + \sum_{n=1}^{\infty} \left[n(n+1)c_{n+1} - (n+2)(n+1)c_{n+2} + (1-n)c_n \right] x^n = 0$$

$c_2 = \frac{1}{2}c_0$ and $c_{n+2} = \dfrac{n(n+1)c_{n+1} + (1-n)c_n}{(n+2)(n+1)}$

$n=1$: $c_3 = \frac{1}{3}c_2 = \frac{1}{3 \cdot 2} c_0$

$n=2$: $c_4 = \dfrac{2c_3}{4} - \dfrac{c_2}{4 \cdot 3} = \dfrac{2}{4 \cdot 3 \cdot 2} c_0 - \dfrac{1}{4 \cdot 3 \cdot 2} c_0$

$$c_4 = \frac{c_0}{4 \cdot 3 \cdot 2}$$

$n=3$: $c_5 = \dfrac{3 \cdot 4 c_4 - 2c_3}{5 \cdot 4} = \dfrac{\frac{1}{2}c_0 - \frac{1}{3}c_0}{5 \cdot 4} = \dfrac{c_0}{5 \cdot 4 \cdot 3 \cdot 2}$

$n=4$: $c_6 = \dfrac{4 \cdot 5 c_5 - 3c_4}{6 \cdot 5} = \dfrac{\frac{c_0}{3 \cdot 2} - \frac{c_0}{4 \cdot 2}}{6 \cdot 5} = \dfrac{c_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$

$c_n = \dfrac{c_0}{n!}$ for $n = 2, 3, 4, \ldots$ Thus

$$y(x) = c_0 + c_1 x + c_0 \sum_{n=2}^{\infty} \frac{x^n}{n!}$$

We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so

$$y(x) = c_0(e^x - x) + c_1 x \qquad e^x = 1 + x + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Initial Conditions give us $c_0 = -2$, $c_1 = 6$, so

$$y(x) = -2e^x + 8x$$