Problem 1:

a) $y'' + y' + y = 1 + e^{-t}$; $y(0) = 3$, $y'(0) = -5$

$$\mathcal{L}(y'' + y' + y) = s^2 Y(s) - s y(0) - y'(0) + s Y(s) - y(0) + Y(s)$$

$$= s^2 Y(s) - 3s + 5 + s Y(s) - 3 + Y(s)$$

$$= Y(s)[s^2 + s + 1] - 3s + 2$$

$$\mathcal{L}(1 + e^{-t}) = \frac{1}{s} + \frac{1}{s+1}$$

Thus, $Y(s)[s^2 + s + 1] - 3s + 2 = \frac{1}{s} + \frac{1}{s+1}$

$$Y(s)[s^2 + s + 1] = \frac{1}{s} + \frac{1}{s+1} + 3s - 2$$

$$Y(s) = \frac{1}{s(s^2+s+1)} + \frac{1}{(s+1)(s^2+s+1)} + \frac{3s}{(s^2+s+1)} - \frac{2}{(s^2+s+1)}$$

part i)

$$\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+s+1}$$

$$A(s^2+s+1) + (Bs + C)(s) = As^2 + As + A + Bs^2 + Cs = 1$$

$A + B = 0$ $\quad B = -1$

$A + C = 0$ $\quad C = -1$

$A = 1$

$$\frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{s+1}{s^2+s+1} = \frac{1}{s} - \frac{s+1}{(s^2 + s + \frac{1}{4} + \frac{3}{4})} =$$

$$= \frac{1}{s} - \frac{s+1}{(s+\frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{s} - \frac{s + \frac{1}{2} + \frac{1}{2}}{(s+\frac{1}{2})^2 + \frac{3}{4}} =$$

$$= \frac{1}{s} - \frac{(s + \frac{1}{2})}{(s+\frac{1}{2})^2 + \frac{3}{4}} - \frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + \frac{3}{4}}$$

$$= \frac{1}{s} - \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \frac{3}{4}} - \frac{1}{2}\sqrt{\frac{4}{3}} \frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2 + \frac{3}{4}}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \frac{3}{4}} - \frac{1}{\sqrt{3}} \frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2 + \frac{3}{4}} \right) =$$

$$= 1 - e^{-\frac{t}{2}} \cos\sqrt{\tfrac{3}{4}}\,t - \frac{1}{\sqrt{3}} e^{-\frac{t}{2}} \sin\sqrt{\tfrac{3}{4}}\,t$$

part ii)

$$\frac{1}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}$$

$$A(s^2+s+1) + (Bs+C)(s+1) = As^2 + As + A + Bs^2 + Bs + Cs + C = 1$$

$A + B = 0 \qquad A = -B \qquad B = -1$

$A + B + C = 0 \qquad C = 0$

$A + C = 1 \qquad A = 1$

$$\frac{1}{s+1} - \frac{s}{s^2+s+1} = \frac{1}{(s+1)(s^2+s+1)}$$

$$\frac{1}{s+1} - \frac{s}{s^2+s+1} = \frac{1}{s+1} - \frac{s}{(s+\frac{1}{2})^2 + \frac{3}{4}} =$$

$$= \frac{1}{s+1} - \frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{s+1} - \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \frac{3}{4}} + \frac{1}{2} \frac{1}{(s+\frac{1}{2})^2 + \frac{3}{4}}$$

$$= \frac{1}{s+1} - \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \frac{3}{4}} + \frac{1}{2}\sqrt{\frac{4}{3}} \frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2 + \frac{3}{4}}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}-\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{3}{4}}+\frac{1}{\sqrt{3}}\frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right)=$$

$$= e^{-t} - e^{-\frac{t}{2}}\cos\sqrt{\tfrac{3}{4}}\,t + \frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\sqrt{\tfrac{3}{4}}\,t$$

part iii)

$$\frac{3s}{s^2+s+1}=\frac{3s}{(s+\frac{1}{2})^2+\frac{3}{4}}=3\left[\frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right]=$$

$$=3\left[\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{3}{4}}-\frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right]=$$

$$=3\left[\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{3}{4}}-\frac{1}{2}\sqrt{\frac{4}{3}}\frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right]=$$

$$\mathcal{L}^{-1}\left(3\left[\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{3}{4}}-\frac{1}{\sqrt{3}}\frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right]\right)=$$

$$3\cdot\left[e^{-\frac{t}{2}}\cos\sqrt{\tfrac{3}{4}}\,t-\frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\sqrt{\tfrac{3}{4}}\,t\right]=$$

$$= 3e^{-\frac{t}{2}}\cos\sqrt{\tfrac{3}{4}}\,t-\sqrt{3}\,e^{-\frac{t}{2}}\sin\sqrt{\tfrac{3}{4}}\,t$$

part iv)

$$\frac{2}{s^2+s+1} = 2\frac{1}{s^2+s+1} = 2\left[\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right] =$$

$$= 2\left[\sqrt{\tfrac{4}{3}}\ \frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right] =$$

$$= \frac{4}{\sqrt{3}}\ \frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

$$\mathcal{L}^{-1}\left(\frac{4}{\sqrt{3}}\ \frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right) = \frac{4}{\sqrt{3}}\, e^{-\frac{t}{2}} \sin\sqrt{\tfrac{3}{4}}\,t$$

Solution:

$$y(t) = 1 - e^{-\frac{t}{2}}\cos\sqrt{\tfrac{3}{4}}\,t - \frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\sqrt{\tfrac{3}{4}}\,t \ +$$

$$e^{-t} - e^{-\frac{t}{2}}\cos\sqrt{\tfrac{3}{4}}\,t + \frac{1}{\sqrt{3}}e^{-t/2}\sin\sqrt{\tfrac{3}{4}}\,t \ +$$

$$3e^{-t/2}\cos\sqrt{\tfrac{3}{4}}\,t - \sqrt{3}\,e^{-t/2}\sin\sqrt{\tfrac{3}{4}}\,t \ -$$

$$\frac{4}{\sqrt{3}}e^{-t/2}\sin\sqrt{\tfrac{3}{4}}\,t \ =$$

$$= 1 + e^{-t} + e^{-t/2}\cos\sqrt{\tfrac{3}{4}}\,t - \frac{7}{\sqrt{3}}e^{-t/2}\sin\sqrt{\tfrac{3}{4}}\,t$$

b) $y'' + 2y' + y = 3\delta(t-1)$; $y(0) = y'(0) = 0$

$$\mathcal{L}(y'' + 2y' + y) = s^2 Y(s) - s\,y(0) - y'(0) + 2s\,Y(s) - 2y(0) + Y(s)$$

$$= s^2 Y(s) + 2s\,Y(s) + Y(s)$$

$$\mathcal{L}(3\,\delta(t-1)) = 3e^{-s}$$

So, $s^2 Y(s) + 2s\,Y(s) + Y(s) = 3e^{-s}$

$$Y(s)\left[s^2 + 2s + 1\right] = 3e^{-s}$$

$$Y(s) = 3e^{-s}\frac{1}{(s^2 + 2s + 1)} =$$

$$= 3e^{-s}\frac{1}{(s+1)^2}$$

$$\mathcal{L}^{-1}\left(Y(s) = 3e^{-s}\frac{1}{(s+1)^2}\right)$$

$$\mathcal{L}^{-1}\left(3e^{-s}\frac{1}{(s+1)^2}\right) = 3\,\mathcal{L}^{-1}\left(e^{-s}\frac{1}{(s+1)^2}\right) =$$

$$= 3\left[\mathcal{U}(t-1)\, t e^{-t}\Big|_{t \to t-1}\right] =$$

$$= 3\,\mathcal{U}(t-1)(t-1)e^{-(t-1)}$$

Thus, $y(t) = 3\,\mathcal{U}(t-1)(t-1)e^{-(t-1)}$

c) $y'' + 4y = \begin{cases} \cos t & t \in [0, \frac{\pi}{2}) \\ 0 & t \in [\frac{\pi}{2}, \infty) \end{cases}$ $\quad y'(0) = y(0) = 0$

$$f(t) = \cos t - \mathcal{U}(t - \tfrac{\pi}{2}) \cos t$$

$$\mathcal{L}(y'' + 4y) = s^2 Y(s) - s y(0) - y'(0) + 4 Y(s)$$
$$= s^2 Y(s) + 4 Y(s)$$
$$= Y(s) [s^2 + 4]$$

$$\mathcal{L}(f(t)) = \mathcal{L}(\cos t - \mathcal{U}(t - \tfrac{\pi}{2}) \cos t) =$$
$$= \mathcal{L}(\cos t) - \mathcal{L}(\mathcal{U}(t - \tfrac{\pi}{2}) \cos t) =$$
$$= \frac{s}{s^2 + 1} - e^{-\frac{\pi}{2}s} \mathcal{L}(\cos(t + \tfrac{\pi}{2})) =$$
$$= \frac{s}{s^2+1} - e^{-\frac{\pi}{2}s} \mathcal{L}(\cos t \cos \tfrac{\pi}{2} - \sin t \sin \tfrac{\pi}{2}) =$$
$$= \frac{s}{s^2+1} - e^{-\frac{\pi}{2}s} \mathcal{L}(-\sin t) =$$
$$= \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s} \mathcal{L}(\sin t) = \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s} \frac{1}{s^2+1}$$

So,

$$Y(s) [s^2 + 4] = \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s} \frac{1}{s^2+1}$$

$$Y(s) = \frac{s}{(s^2+1)(s^2+4)} + e^{-\frac{\pi}{2}s} \frac{1}{(s^2+1)(s^2+4)}$$

part i)

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{s^2+4}$$

$$(As+B)(s^2+4) + (Cs+D)(s^2+1) =$$

$$= As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D = s$$

$$A + C = 0 \qquad B + D = 0 \qquad 4A + C = 1 \qquad 4B + D = 0$$

$$\longrightarrow \quad A = -C \qquad B + D = 0 \qquad 4A - A = 1 \qquad 4B + D = 0$$

$$\longrightarrow \quad A = -C \qquad B + D = 0 \qquad 3A = 1 \qquad 4B + D = 0$$

$$A = \frac{1}{3} \qquad C = -\frac{1}{3} \qquad B + D = 0 \qquad 4B + D = 0$$

$$\longrightarrow \quad \begin{array}{r} 4B + D = 0 \\ (-)\ B + D = 0 \\ \hline 3B = 0 \end{array} \qquad B = 0 \qquad D = 0$$

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{3}s}{s^2+1} - \frac{\frac{1}{3}s}{s^2+4}\right) =$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) =$$

$$= \frac{1}{3}\cos t - \frac{1}{3}\cos 2t$$

part iii)

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\left.\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ 4A + C &= 0 \\ 4B + D &= 1 \end{aligned}\right\} \quad \begin{aligned} 3B &= 1 \\ B &= \tfrac{1}{3} \\ D &= -\tfrac{1}{3} \end{aligned} \quad \begin{aligned} A &= 0 \\ C &= 0 \end{aligned}$$

$$\mathcal{L}^{-1}\left(e^{-\frac{\pi}{2}s}\left(\frac{\frac{1}{3}}{(s^2+1)} - \frac{\frac{1}{3}}{(s^2+4)}\right)\right) =$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)\right) =$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1} - \frac{1}{2}\frac{2}{s^2+4}\right)\right) =$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(e^{-\frac{\pi}{2}s}\frac{1}{s^2+1} - \frac{1}{2}e^{-\frac{\pi}{2}s}\frac{2}{s^2+4}\right) =$$

$$= \frac{1}{3}\left[\mathcal{U}(t-\tfrac{\pi}{2})\sin(t-\tfrac{\pi}{2}) - \frac{1}{2}\mathcal{U}(t-\tfrac{\pi}{2})\sin 2(t-\tfrac{\pi}{2})\right]$$

$$= \frac{1}{3}\mathcal{U}(t-\tfrac{\pi}{2})\sin(t-\tfrac{\pi}{2}) - \frac{1}{6}\mathcal{U}(t-\tfrac{\pi}{2})\sin 2(t-\tfrac{\pi}{2})$$

Solution:

$$y(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t + \frac{1}{3}\mathcal{U}(t-\tfrac{\pi}{2})\sin(t-\tfrac{\pi}{2})$$
$$- \frac{1}{6}\mathcal{U}(t-\tfrac{\pi}{2})\sin 2(t-\tfrac{\pi}{2})$$

Problem 2)

a) $\mathcal{L}^{-1}\left(\frac{1}{(s-a)^n}\right) = \mathcal{L}^{-1}\left(\frac{1}{(n-1)!}\,\frac{(n-1)!}{(s-a)^n}\right)$

$= \mathcal{L}^{-1}\left(\frac{1}{m!}\,\frac{m!}{(s-a)^{m+1}}\right)$ where $m = (n-1)$

$= \frac{1}{m!}\,\mathcal{L}^{-1}\left(\frac{m!}{(s-a)^{m+1}}\right) =$

$= \frac{1}{m!}\,t^m\,e^{at} = \frac{1}{(n-1)!}\,t^{n-1}\,e^{at}$

b) $\mathcal{L}^{-1}\left(\frac{e^{-as}}{s}\right) = \mathcal{U}(t-a)$

c) $\mathcal{L}^{-1}\left(e^{-as}\,\frac{1}{(s-b)^2+K^2}\right) = \mathcal{U}(t-a)\,\mathcal{L}^{-1}\left(\frac{1}{(s-b)^2+K^2}\right)\Big|_{t\to t-a}$

$= \mathcal{U}(t-a)\,\frac{1}{K}\,\mathcal{L}^{-1}\left(\frac{K}{(s-b)^2+K^2}\right)\Big|_{t\to t-a} =$

$= \mathcal{U}(t-a)\,\frac{1}{K}\,e^{bt}\,\operatorname{Sinh} Kt\,\Big|_{t\to t-a} =$

$= \mathcal{U}(t-a)\,\frac{1}{K}\,e^{b(t-a)}\,\operatorname{Sinh} K(t-a)$

d) $\mathcal{L}(t\cdot f(t)) = (-1)\,\frac{d}{ds}\,\mathcal{L}(f(t)) =$

$= -\frac{d}{ds}\,\mathcal{L}(f(t)) = (-1)\,\frac{d}{ds}\,F(s)$

Problem 3)

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \, dt =$$

$$= \int_0^\infty e^{t(a-s)} \, dt = \frac{1}{(a-s)} e^{t(a-s)} \Big|_0^\infty =$$

(Assuming $a-s<0$)

$$\frac{1}{(a-s)} e^{t(a-s)} \Big|_0^\infty = 0 - \frac{1}{(a-s)} e^0 =$$

$$= \frac{-1}{(a-s)} = \frac{1}{(s-a)}$$

Problem 4)

c) $f(t) = \mathcal{U}(t-0) + 4\,\mathcal{U}(t-3) - 3\,\mathcal{U}(t-4)$

b) $f(t) = \begin{cases} 0 & t \in [0,2) \\ 2(t-3)^2 & t \in [2,4) \\ 0 & t \in [4,\infty) \end{cases} =$

$$= \mathcal{U}(t-2)\,2(t-3)^2 - \mathcal{U}(t-4)\,2(t-3)^2$$

Problem 5)

Given: $\mathcal{L}(\mathcal{U}(t-a)f(t-a)) = e^{-as}\mathcal{L}(f(t))$

$$\mathcal{L}^{-1}(\mathcal{L}(\mathcal{U}(t-a)f(t-a))) = \mathcal{L}^{-1}(e^{-as}\mathcal{L}(f(t)))$$

$$\mathcal{U}(t-a)f(t-a) = \mathcal{L}^{-1}(e^{-as}F(s))$$

So, $\mathcal{L}^{-1}(e^{-as}F(s)) = \mathcal{U}(t-a)f(t-a) =$

$$= \mathcal{U}(t-a)\mathcal{L}^{-1}(F(s))\Big|_{t \to t-a}$$

where $F(s) = \mathcal{L}(f(t))$