Math 527
HW # 7
Answer Key

#1

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

Int by parts: let $u = t^n$ $\quad dv = e^{-st} dt$

$du = n t^{n-1}$ $\quad v = \frac{-1}{s} e^{-st}$

$$\int_0^\infty e^{-st} t^n \, dt = \frac{t^n}{s} e^{-st} \Big|_0^\infty - \int_0^\infty -\frac{n t^{n-1}}{s} e^{-st} \, dt =$$

$$= 0 + \int_0^\infty \frac{n t^{n-1}}{s} e^{-st} \, dt = \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt =$$

$$= \frac{n}{s} \mathcal{L}(t^{n-1}) \quad \text{by def. of } \mathcal{L}\text{-transform.}$$

Then,

we know $\mathcal{L}(t) = \frac{1}{s^2}$

$\mathcal{L}(t^2) = \frac{2}{s} \mathcal{L}(t) = \frac{2}{s} \frac{1}{s^2} = \frac{2}{s^3}$ by above argument.

And, furthermore,

$$\mathcal{L}(t^3) = \frac{3}{s} \mathcal{L}(t^2) = \frac{3}{s} \cdot \frac{2}{s^3} = \frac{6}{s^4} = \frac{3!}{s^4}$$

$$\mathcal{L}(t^4) = \frac{4}{s} \mathcal{L}(t^3) = \frac{4}{s} \cdot \frac{6}{s^4} = \frac{24}{s^5} = \frac{4!}{s^5}$$

etc...

This implies that

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

#2

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\, dt =$$

$$= \int_0^\infty e^{(a-s)t}\, dt = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty =$$

$$= \frac{1}{a-s} \left[e^{(a-s)t} \Big|_0^\infty \right] =$$

$$= \frac{1}{a-s} \left[0 - 1 \right]$$ assuming that $a-s < 0$

Then, $\frac{1}{a-s}(-1) = \frac{1}{s-a}$

#3

$$\mathcal{L}\{\sinh kt\} = \mathcal{L}\left\{\frac{e^{kt} - e^{-kt}}{2}\right\} =$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{kt} - e^{-kt}\}\right] =$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{kt}\} - \mathcal{L}\{e^{-kt}\}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{s-k} - \frac{1}{s+k}\right]$$ by problem #2.

Then, $\frac{1}{2}\left[\frac{1}{s-k} - \frac{1}{s+k}\right] =$

$$= \frac{1}{2}\left[\frac{s+k}{(s+k)(s-k)} - \frac{s-k}{(s+k)(s-k)}\right] =$$

$$= \frac{1}{2}\left[\frac{2k}{(s+k)(s-k)}\right] = \frac{k}{s^2 - k^2}$$

#4

$$\mathcal{L}\{\cosh kt\} = \mathcal{L}\left\{\frac{e^{kt}+e^{-kt}}{2}\right\} =$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{kt}+e^{-kt}\}\right] =$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right] =$$

$$= \frac{1}{2}\left[\frac{s+k}{(s+k)(s-k)} + \frac{s-k}{(s+k)(s-k)}\right] =$$

$$= \frac{1}{2}\left[\frac{2s}{(s+k)(s-k)}\right] = \frac{s}{s^2-k^2}$$

#5

Ansatz procedure:

- let $y = e^{\lambda t}$

$$y'' - y' - 2y = e^{\lambda t}(\lambda^2 - \lambda - 2) = 0$$

Then, $\lambda^2 - \lambda - 2 = 0$

$$(\lambda - 2)(\lambda + 1) = 0 \quad \text{and} \quad \lambda = 2, -1$$

- homogeneous soln:

$$y_h = C_1 e^{2t} + C_2 e^{-t}$$

apply I.C.'s:

$$y(t) = C_1 e^{2t} + C_2 e^{-t}$$

$$y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$$

$y(0) = 1$, $y'(0) = 0$

$$y(0) = C_1 + C_2 = 1$$

$$y'(0) = 2C_1 - C_2 = 0$$

Then, $C_1 = \frac{1}{3}$, $C_2 = \frac{2}{3}$

Soln:

$$y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

Laplace procedure:

$$\mathcal{L}(y'') = s^2 \cdot Y(s) - s\,y(0) - y'(0)$$

$$= s^2 Y(s) - s(1) - 0$$

$$= s^2 Y(s) - s$$

$$\mathcal{L}(y') = s \cdot Y(s) - y(0)$$

$$= s\,Y(s) - 1$$

$$\mathcal{L}(y) = Y(s)$$

substituting into D.E.:

$$\mathcal{L}(y'' - y' - 2y) = s^2 Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

factor $Y(s)$:

$$Y(s)\left[s^2 - s - 2\right] - s + 1 = 0$$

$$Y(s)\left[s^2 - s - 2\right] = s - 1$$

$$Y(s) = \frac{s-1}{s^2 - s - 2}$$

P.F.D:

$$\frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$\frac{A(s+1)}{(s+1)(s-2)} + \frac{B(s-2)}{(s+1)(s-2)} = \frac{s-1}{(s-2)(s+1)}$$

$$A(s+1) + B(s-2) = s-1$$

$$As + A + Bs - 2B = s - 1$$

$$A + B = 1 \qquad\qquad 2A + 2B = 2$$

$$A - 2B = -1 \quad \text{or} \quad - \;\; A - 2B = -1$$

$$3A = 1$$

$$A = \frac{1}{3}, \quad B = \frac{2}{3}$$

So,

$$Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

Inverse Transform:

$$\mathcal{L}^{-1}\left[Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1} \right]$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{\frac{2}{3}}{s+1}\right) =$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) =$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Soln:

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$