Math. 527
HW #5
Answer Key

#1.

Given: $e^{ix} = \cos x + i \sin x$; $e^{-ix} = \cos x - i \sin x$

Show: $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and

$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

<u>Proof:</u>

$$\frac{e^{ix} + e^{-ix}}{2} = \frac{\cos x + i \sin x + \cos x - i \sin x}{2} =$$

$$= \frac{2 \cos x}{2} = \cos x$$

$$\frac{e^{ix} - e^{-ix}}{2i} = \frac{\cos x + i \sin x - \cos x + i \sin x}{2i} =$$

$$= \frac{2i \sin x}{2i} = \sin x$$

#2

a) Show: $C_1 \cos \omega t + C_2 \sin \omega t$ can be written in the form $A \sin(\omega t + \phi)$

Proof:

$$A \sin(\omega t + \phi) = A [\sin \omega t \cos \phi + \cos \omega t \sin \phi] =$$

$$= A \sin \omega t \cos \phi + A \cos \omega t \sin \phi$$

Let $C_1 = A \sin \phi$ and $C_2 = A \cos \phi$

Then $A \sin \omega t \cos \phi + A \cos \omega t \sin \phi =$

$$= C_2 \sin \omega t + C_1 \cos \omega t$$

$\therefore$, $A \sin(\omega t + \phi) = C_1 \cos \omega t + C_2 \sin \omega t$

b) $$\frac{C_1}{C_2} = \frac{A \sin \phi}{A \cos \phi} = \frac{\sin \phi}{\cos \phi} = \tan \phi$$

$\therefore$, $\phi = \tan^{-1}\left(\frac{C_1}{C_2}\right)$

$A \sin \phi = C_1 \Rightarrow A = \dfrac{C_1}{\sin \phi} \Rightarrow$

$$\Rightarrow A = \frac{C_1}{\sin\left(\tan^{-1}\left(\frac{C_1}{C_2}\right)\right)}$$

b) Cont'd.

Similarly

$$A \cos\phi = C_2 \Rightarrow A = \frac{C_2}{\cos\phi} \Rightarrow$$

$$\Rightarrow A = \frac{C_2}{\cos\left(\tan^{-1}\left(\frac{C_1}{C_2}\right)\right)}$$

c) $C_1 = A \sin\phi$, $C_2 = A\cos\phi$

as previously defined.

#3

Given: $my'' + \beta y' + ky = 0$

Solution:

$$\frac{-\frac{\beta}{m} \pm \sqrt{\left(\frac{\beta}{m}\right)^2 - 4(1)\left(\frac{k}{m}\right)}}{2}$$

a) Overdamped:

$$\left(\frac{\beta}{m}\right)^2 - 4\frac{k}{m} > 0$$

$$\frac{\beta^2}{m^2} > 4\frac{k}{m}$$

$$\beta^2 > 4km$$

$$\beta > 2\sqrt{km} \quad \text{since } \beta > 0$$

b) Critically damped:

$$\left(\frac{\beta}{m}\right)^2 - 4\frac{k}{m} = 0$$

$$\frac{\beta^2}{m^2} = 4\frac{k}{m}$$

$$\beta^2 = 4km$$

$$\beta = 2\sqrt{km}$$

c) Underdamped:

$$\left(\frac{\beta}{m}\right)^2 - 4\frac{k}{m} < 0$$

$$\frac{\beta^2}{m^2} < 4\frac{k}{m}$$

$$\beta^2 < 4km$$

$$\beta < 2\sqrt{km}$$

#4 $y'' + y' - 2y = 2x$

homog. solution:

let $y = e^{\lambda x}$

$y'' + y' - 2y = 0$

$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2e^{\lambda x} = 0$

$\lambda^2 + \lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$

$\lambda_1 = -2$

$\lambda_2 = 1$

$y_1(x) = e^{-2x}$

$y_2(x) = e^{x}$

part. solution:

let $y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$\Rightarrow 0 + A - 2Ax - 2B = 2x$

$-2A = 2$

$A = -1$

$A - 2B = 0$

$-1 - 2B = 0$

$-2B = 1$

$B = \frac{-1}{2}$

Thus, $y_p = -x - \frac{1}{2}$

$$\boxed{y_{gen}(x) = C_1 e^{-2x} + C_2 e^{x} - x - \frac{1}{2}}$$

Check:

$$y'(x) = -2C_1 e^{-2x} + C_2 e^{x} - 1$$

$$y''(x) = 4C_1 e^{-2x} + C_2 e^{x}$$

subst:

$$4C_1 e^{-2x} + C_2 e^{x} + (-2C_1 e^{-2x} + C_2 e^{x} - 1)$$
$$-2\left(C_1 e^{-2x} + C_2 e^{x} - x - \frac{1}{2}\right) =$$

$$= 4C_1 e^{-2x} + C_2 e^{x} - 2C_1 e^{-2x} + C_2 e^{x} - 1$$
$$- 2C_1 e^{-2x} - 2C_2 e^{x} + 2x + 1 =$$

$$= 2x$$

#5 $y'' + 4y = x^2 + e^x$

homog. soln:

$\lambda^2 + 4 = 0$

$\lambda^2 = -4$

$\lambda = \pm 2i$

$y_1(x) = \sin 2x$

$y_2(x) = \cos 2x$

part. soln:

let $y_p = Ax^2 + Bx + C + De^x$

$y_p' = 2Ax + B + De^x$

$y_p'' = 2A + De^x$

$\Rightarrow 2A + De^x + 4(Ax^2 + Bx + C + De^x) =$

$= 2A + De^y + 4Ax^2 + 4Bx + 4C + 4De^x = x^2 + e^x$

$4A = 1$

$A = \frac{1}{4}$

$4B = 0$

$B = 0$

$2A + 4C = 0$

$\frac{1}{2} + 4C = 0$

$4C = -\frac{1}{2}$

$C = -\frac{1}{8}$

$5D = 1$

$D = \frac{1}{5}$

$y_p = \frac{1}{4}x^2 - \frac{1}{8} + \frac{1}{5}e^x$

$y_{gen}(x) = C_1 \sin 2x + C_2 \cos 2x + \frac{1}{4}x^2 - \frac{1}{8} + \frac{1}{5}e^x$

Check:

$$y'(x) = 2C_1 \cos 2x - 2C_2 \sin 2x + \frac{1}{2}x + \frac{1}{5}e^x$$

$$y''(x) = -4C_1 \sin 2x - 4C_2 \cos 2x + \frac{1}{2} + \frac{1}{5}e^x$$

Subst:

$$-4C_1 \sin 2x - 4C_2 \cos 2x + \frac{1}{2} + \frac{1}{5}e^x +$$

$$4C_1 \sin 2x + 4C_2 \cos 2x + x^2 - \frac{1}{2} + \frac{4}{5}e^x =$$

$$= x^2 + e^x$$

#6 $y'' + 4y = 3 \sin 2x$

homog. soln:

$$y_1(x) = \sin 2x$$

$$y_2(x) = \cos 2x$$

part. soln:

$$y_p = Ax \sin 2x + Bx \cos 2x$$

$$y_p' = A \sin 2x + 2Ax \cos 2x + B \cos 2x - 2Bx \sin 2x$$

$$y_p'' = 2A \cos 2x + 2A \cos 2x - 4Ax \sin 2x - 2B \sin 2x - 2B \sin 2x - 4Bx \cos 2x$$

$$\Rightarrow \quad 4A \cos 2x - 4Ax \sin 2x - 4B \sin 2x - 4Bx \cos 2x + 4Ax \sin 2x + 4Bx \cos 2x = 3 \sin 2x$$

$$4A \cos 2x - 4B \sin 2x = 3 \sin 2x$$

$$4A = 0$$

$$A = 0$$

$$y_p = -\frac{3}{9} x \cos 2x$$

$$-4B = 3$$

$$B = \frac{-3}{4}$$

$$y_{gen}(x) = C_1 \sin 2x + C_2 \cos 2x - \frac{3}{4} x \cos 2x$$

Check:

$$y'(x) = 2C_1 \cos 2x - 2C_2 \sin 2x - \frac{3}{4} \cos 2x + \frac{6}{4} x \sin 2x$$

$$y''(x) = -4C_1 \sin 2x - 4C_2 \cos 2x + \frac{6}{4} \sin 2x +$$

$$\frac{6}{4} \sin 2x + \frac{12}{4} x \cos 2x$$

subst.:

$$-4C_1 \sin 2x - 4C_2 \cos 2x + \frac{12}{4} \sin 2x + \frac{12}{4} x \cos 2x$$

$$+ 4C_1 \sin 2x + 4C_2 \cos 2x - \frac{12}{4} x \cos 2x =$$

$$\frac{12}{4} \sin 2x = 3 \sin 2x$$