problem #5

Given: $x' = Ax$

Assume: $\underline{x} = \underline{V} e^{\lambda t}$

$$\underline{x}' = \underline{V} \lambda e^{\lambda t}$$

substitute into $x' = Ax$

Thus, $\underline{V} \lambda e^{\lambda t} = A \underline{V} e^{\lambda t}$

$$\Rightarrow \underline{V} \lambda = A \underline{V}$$

$$\Rightarrow A\underline{V} - V\lambda = 0$$

$$\Rightarrow (A - \lambda I)\underline{V} = 0$$

So, if $(A - \lambda I)\underline{V} = 0$, then $\det(A - \lambda I) = 0$

in order to find a non-trivial solution vector $\underline{V}$.

Thus, $\det(A - \lambda I) = 0$

problem #6

$$\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = 4x - 6x \longrightarrow \underset{\underline{X'}}{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}} = \underset{A}{\begin{pmatrix} 1 & 2 \\ 4 & -6 \end{pmatrix}} \underset{\underline{X}}{\begin{pmatrix} x \\ y \end{pmatrix}}$$

let $\underline{X} = \underline{V} e^{\lambda t}$

$\underline{X}' = \underline{V} \lambda e^{\lambda t}$

Substitute into $\underline{X}' = A\underline{X}$

Then, $\underline{V} \lambda e^{\lambda t} = A \underline{V} e^{\lambda t}$

$\Rightarrow \underline{V}\lambda = A\underline{V} \Rightarrow A\underline{V} - \lambda \underline{V} = 0$

$\Rightarrow (A - \lambda \underline{I}) \underline{V} = 0$

Find $(A - \lambda \underline{I})$:

$$(A - \lambda \underline{I}) = \begin{pmatrix} 1 & 2 \\ 4 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 4 & -6-\lambda \end{pmatrix}$$

Set $\det(A - \lambda \underline{I}) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & -6-\lambda \end{vmatrix} = (1-\lambda)(-6-\lambda) - 8 = -6 - \lambda + 6\lambda + \lambda^2 - 8 =$$

$$= \lambda^2 + 5\lambda - 14 = 0$$

Factor:

$$(\lambda+7)(\lambda-2)=0$$

$$\Rightarrow \lambda_1 = -7 \quad , \lambda_2 = 2$$

For $\lambda_1 = -7$

$$(A - \lambda I) = \begin{pmatrix} 8 & 2 \\ 4 & 1 \end{pmatrix}$$

$$(A - \lambda I)\underline{V} = 0$$

$$\Rightarrow \begin{pmatrix} 8 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 4V_1 + V_2 = 0$$

$$V_2 = -4V_1 \quad \text{or} \quad \underline{V} = \begin{pmatrix} V_1 \\ -4V_1 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Thus, $\underline{X}_1 = \underline{V}e^{\lambda t} = C_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-7t}$

For $\lambda_2 = 2$

$$\begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -V_1 + 2V_2 = 0$$

$$\Rightarrow \underline{V} = \begin{pmatrix} 2V_2 \\ V_2 \end{pmatrix} = C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus, $\underline{X}_2 = C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}$

General Solution:

$$\underline{X} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-7t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

Problem # 7

$$\frac{dx}{dt} = -x + y$$

$$\frac{dy}{dt} = x + 2y + z \longrightarrow X' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} X$$

$$\frac{dz}{dt} = 3y - z$$

$$(A - \lambda I) = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{pmatrix}$$

$$(A - \lambda I)\underline{V} = 0$$

Set $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix} =$$

$$= -1-\lambda \begin{vmatrix} 2-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & -1-\lambda \end{vmatrix} + 0 =$$

$$= (-1-\lambda)((2-\lambda)(-1-\lambda)-3) - (-1-\lambda) =$$

$$= (-1-\lambda)(-2-2\lambda+\lambda+\lambda^2-3) + (1+\lambda) =$$

$$= (-1-\lambda)(\lambda^2-\lambda-5) + (1+\lambda) =$$

$$= (-1-\lambda)(\lambda^2-\lambda-5-1) = (-1-\lambda)(\lambda^2-\lambda-6) = 0$$

$$(-1-\lambda)(\lambda^2-\lambda-6) = (-1-\lambda)(\lambda-3)(\lambda+2) = 0$$

Thus, $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = -2$

For $\lambda_1 = -1$,

$$(A-\lambda I)\underline{V} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right) \xrightarrow[\text{Steps.}]{\text{Gaussian Elim.}} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, $V_2 = 0$, $V_1 + V_3 = 0$

$$\underline{V} = \begin{pmatrix} -V_3 \\ 0 \\ V_3 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

And, $\underline{X_1} = C_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}$

For $\lambda_2 = 3$,

$$(A - \lambda I)\underline{V} = \begin{pmatrix} -4 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -4 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or}$$

$$\left(\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) \xrightarrow{R_1, R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{4R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) \xrightarrow{R_2 + R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, $V_1 - \frac{1}{3}V_3 = 0$, $V_2 - \frac{4}{3}V_3 = 0$

$V_1 = \frac{1}{3}V_3$, $V_2 = \frac{4}{3}V_3$

$$\underline{V} = \begin{pmatrix} \frac{1}{3} V_3 \\ \frac{4}{3} V_3 \\ V_3 \end{pmatrix} = C_2 \begin{pmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{pmatrix} = \hat{C}_2 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

And, $\underline{X}_2 = \hat{C}_2 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} e^{3t}$

Finally, for $\lambda_3 = -2$

$$(A - \lambda I)\underline{V} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, $V_1 + V_2 = 0 \quad , \quad 3V_2 + V_3 = 0$

$$V_1 = -V_2 \quad , \quad 3V_2 = -V_3$$

$$V_1 = -V_2 \quad , \quad V_3 = -3V_2$$

$$\underline{V} = \begin{pmatrix} -V_2 \\ V_2 \\ -3V_2 \end{pmatrix} = C_3 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$$

And,
$$\underline{X}_3 = C_3 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} e^{-2t}$$

General Solution:

$$\underline{X} = C_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} e^{3t} + C_3 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} e^{-2t}$$

problem # 8

I.C.'s: $x(0) = 1$, $y(0) = 0$, $z(0) = 2$ or

$$\underline{X}(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Gen. Soln: $\underline{X} = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} e^{-2t}$

$$x(0) = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Thus,

$$\begin{aligned} -c_1 + c_2 - c_3 &= 1 \\ 4c_2 + c_3 &= 0 \\ c_1 + 3c_2 + 3c_3 &= 2 \end{aligned}$$

or

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 4 & 1 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{or} \quad \left(\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 0 & 4 & 1 & 0 \\ 1 & 3 & 3 & 2 \end{array}\right)$$

perform Gaussian Elimination to solve for c_1, c_2, c_3.

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 0 & 4 & 1 & 0 \\ 1 & 3 & 3 & 2 \end{array}\right) \xrightarrow{(-1)R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 4 & 1 & 0 \\ 1 & 3 & 3 & 2 \end{array}\right) \xrightarrow{R_3 - R_1}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 4 & 1 & 0 \\ 0 & 4 & 2 & 3 \end{array}\right) \xrightarrow{\frac{1}{4}R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 4 & 2 & 3 \end{array}\right) \xrightarrow{-4R_2 + R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & -1 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{-\frac{5}{4}R_3 + R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{19}{4} \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{-\frac{1}{4}R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{19}{4} \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Thus, $C_1 = -\frac{19}{4}$, $C_2 = -\frac{3}{4}$, $C_3 = 3$

$$\underline{X} = -\frac{19}{4}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} - \frac{3}{4}\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} e^{3t} + 3\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} e^{-2t}$$

or

$$\underline{X} = \begin{pmatrix} \frac{19}{4} \\ 0 \\ -\frac{19}{4} \end{pmatrix} e^{-t} - \begin{pmatrix} \frac{3}{4} \\ 3 \\ \frac{9}{4} \end{pmatrix} e^{3t} + \begin{pmatrix} -3 \\ 3 \\ 9 \end{pmatrix} e^{-2t}$$