Homework #3 **IAM 961, UNH fall 2011**
Due Friday, Oct 28th in lecture

1. Write a function `A = randommatrix(m, kappa)` that returns an $m \times m$ random matrix with condition number κ and exponentially graded singular values (i.e. $\sigma_1/\sigma_m = \kappa$ and $\sigma_n/\sigma_{n+1} \approx$ const). You can do this by generalizing the Matlab code on pg 65 of Trefethen & Bau.

2. Write functions for computing the QR decomposition of a matrix via
(a) classical Gram-Schmidt orthogonalization,
(b) modified Gram-Schmidt orthogonalization, and
(c) Householder triangularization.

Pseudocode for these algorithms is given in Trefethen & Bau lectures 7, 8 and 10, but please try to write implement the algorithm on your own, starting from the equations and the textual description of the algorithms, rather than just copying the pseudocode. Use whatever programming language you like. A scripted high-level language with built-in matrix functionality (e.g. Matlab, Octave, Python) will probably be easiest and most revealing. I suggest the following function names: `QRcgs, QRmgs, QRhouse`, and function signatures of the form `[Q,R] = QRcgs(A)`.

3. Write an `x = backsolve(R,b)` function for backsolving the system $Rx = b$ for upper-triangular matrices R.

4. Write an `x = solve(A,b)` function that solves $Ax = b$ via QR decomposition, using the `QRcgs` and `backsolve` sunctions from problems 1 and 2. For fixed m (say $m = 128$), test your `solve` function as follows. Generate a random A and a random x, and use those to compute $b = Ax$. Then let $\hat{x}$ be the result of the `solve(A,b)` function. Do this a few hundred or a few thousand times for κ in the range $1 \leq \kappa \leq 10^{16}$. Produce scatter plots of (a) the error norm $\|\hat{x} - x\|$ versus κ, and (b) the residual norm $\|r\| = \|A\hat{x} - b\|$ versus κ. Use log-log axes for the plots. Repeat with `QRmgs` and `QRhouse`.

5. Comment on your results. Canyou give an estimate of the upper bound of the error norm and the residual norm as a function of κ? What can you explain about the scatter plots based on the algorithms and their implementation in finite-precision arithmetic? Play numerical detective by examining how accurately your QR algorithms produce QR decompositions, and relating that to the results shown in the three scatter plots.

6. Repeat problem 4 for a different value of m (perhaps $m = 256$). (You don't need to turn in these plots.) Does the dimensionality of the matrix have any effect on how your answer for problem 5?