1) True or false: A, B unitarily equiv $\iff$ A, B have same sing vals
(a) (b)

<u>first look at (a) $\overset{?}{\Rightarrow}$ (b)</u>

A, B unitarily equiv means $A = QBQ^*$ for some unitary Q

Let SVD of B be $B = U\Sigma V^*$

Then $A = QU\Sigma V^*Q^*$
$= (QU)\Sigma(QV)^*$

This is the SVD of A, since QU and QV are unitary + Σ is diag

So A has same sing vals $\sigma_i = \Sigma_{ii}$ as B. So (a) $\Rightarrow$ (b)

So unitary equivalence <u>does</u> imply same sing values

now check (a) $\overset{?}{\Leftarrow}$ (b)

Note that Hermitianness is preserved under transformations of the form $B \to QBQ^*$, since $(QBQ^*)^* = QB^*Q^* = QBQ^*$.
So we can prove (b) $\not\Rightarrow$ (a) by finding a hermitian B and a nonhermitian A with the same singular values. Do this by construction with SVD and same Σ

Let $B = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ hermitian

Let $A = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&-1\\1&0\end{bmatrix}$ nonhermitian

So A + B are <u>not</u> unitarily equivalent, yet they have the same eigenvalues. So (b) $\not\Rightarrow$ (a), and we have

unitarily equiv $\Rightarrow$ same eigenvals

same eigenvals $\not\Rightarrow$ unitarily equiv.

2. Show that any real-valued matrix has a real SVD.

For real $A \in \mathbb{R}^{m \times m}$, $A^T A$ is real symmetric and thus has a complete set of real, orthogonal eigenvectors, and non-negative eigenvalues

$$A^T A v_i = \lambda_i v_i \quad , \quad v_i^T v_j = \delta_{ij} \quad \text{for } i,j \in [1,m]$$

Let $\sigma_i = \sqrt{\lambda_i}$

And let $u_i = \frac{1}{\sigma_i} A v_i$ $\qquad \sigma_i u_i = A v_i$

Note
$$\begin{aligned} u_i^T u_j &= \frac{1}{\sigma_i} v_i^T A^T \frac{1}{\sigma_j} A v_j \\ &= \frac{1}{\sigma_i \sigma_j} v_i^T A^T A v_j \\ &= \frac{1}{\sigma_i \sigma_j} v_i^T \sigma_j^2 v_j \\ &= \frac{\sigma_j}{\sigma_i} \delta_{ij} = \delta_{ij} \qquad i,j \in [1,m] \end{aligned}$$

So u_i are real, complete, and orthogonal as well

Let $V = [v_1 | \cdots | v_m]$ $\quad U = [u_1 | \cdots | u_m]$ $\quad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \ \sigma_m \end{bmatrix}$

Then
$$U\Sigma = AV \quad (\text{from } \sigma_i u_i = A v_i)$$
$$U \Sigma V^T = A \cancel{V V^T}^{\to I}$$

So $A = U \Sigma V^T$ is a real-valued SVD of A

3. Show that the nonzero sing vals of A are the square roots of the nonzero eigvals of A^*A or AA^* for $A \in \mathbb{C}^{m \times n}$

Let $A = U\Sigma V^*$ be the SVD of A

$$A^*A = (U\Sigma V^*)^* U\Sigma V^*$$

$$= V\Sigma U^* U\Sigma V^*$$

($\Sigma = \Sigma^*$ b/c Σ is real diagonal)

$$A^*A = V\Sigma^2 V^*$$

$$A^*AV = V\Sigma^2$$

So $(A^*A)v_i = \sigma_i^2 v_i$ where v_i is i^{th} col of V

σ_i^2 is i^{th} diag elem of Σ^2

$\therefore$ the squares of the singular values of A are eigenvalues of A^*A.

Analysis is similer for AA^*.

The reason to specify nonzero singvals & nonzero eigenvals is that for $m>n$, A will have n singular values, and AA^* will have $m>n$ eigenvalues, with the last $m-n$ eigenvalues equaling zero. (similarly for $m<n$ and A^*A). Specifying nonzero assures a 1-to-1 correspondence.

5) Are there flaws in Trefethen + Bau's proof of the uniqueness of the SVD.

First, in the middle of page 30, the demonstration that $\|Av_2\|_2 = \sigma_1$

via $\quad w = cv_1 + sv_2 \qquad\qquad c = \cos\theta, \quad s = \sin\theta$

$$Aw = cAv_1 + sAv_2$$

$$\|Aw\|_2^2 = c^2\|Av_1\|_2 + s^2\|Av_2\|_2$$

$$\sigma_1^2 = c^2\sigma_1^2 + s^2\|Av_2\|_2$$

$$\cancel{(1-c^2)}\sigma_1^2 = s^2\|Av_2\|_2 \qquad s^2 = 1-c^2 \neq 0 \text{ since } w \text{ is linly indpt of } v_1$$

$$\sigma_1^2 = \|Av_2\|_2$$

assumes without proof that $Av_1 \perp Av_2$. More carefully,

$$\|Aw\|_2^2 = (Aw)^* Aw$$

$$= (c\, v_1^*A^* + s v_2^* A^*)(cAv_1 + sAv_2)$$

$$= c^2 v_1^*A^*Av_1 + s^2 v_2^*A^*Av_2 + cs(v_1^*A^*Av_2 + v_2^*A^*Av_1)$$

$$\|Aw\|_2^2 = c^2\|Av_1\|_2 + s^2\|Av_2\|_2 + cs\left[(Av_1)^*Av_2 + (Av_2)^*Av_1\right]$$

We need to prove $(Av_1)^*(Av_2) = (Av_2)^*Av_1 = 0$, ie $Av_1 \perp Av_2$.

We can prove this by appealing back to the factorization

$$U_1^* A V_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & B \end{bmatrix} \quad \text{or} \quad A = U_1 \begin{bmatrix} \sigma_1 & 0 \\ 0 & B \end{bmatrix} V_1^*$$

that was established in the existence portion of the proof. Here v_1 is the first column of V_1, and the remaining cols v_j of V_1 are an arbitrary orthonormal extension of v_1 to an orthonormal basis of $\mathbb{C}^m$. The first column of U_1 is

$$u_1 = \begin{cases} \frac{1}{\sigma_1} Av_1 & \sigma_1 \neq 0 \\ \text{arbitrary unit vec} & \sigma_1 = 0 \end{cases}$$

and the remaining cols are an arb. orthonormal extension, as for V_1. Denote the extension of v_1 by

$$V_1 = \left[v_1 \middle| \hat{V}_1 \right]$$

From $A = U_1 \begin{bmatrix} \sigma_1 & 0 \\ 0 & B \end{bmatrix} V_1^*$, $AV_1 = U_1 \begin{bmatrix} \sigma_1 & 0 \\ 0 & B \end{bmatrix}$

and $V_1^* A^* A V_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & B^* \end{bmatrix} \cancel{U_1^* U_1} \begin{bmatrix} \sigma_1 & 0 \\ 0 & B \end{bmatrix}$

$$\left[\begin{array}{c} v_1^* \\ \hline \hat{V}_1^* \end{array}\right] A^* A \left[\begin{array}{c|c} v_1 & \hat{V}_1 \end{array}\right] = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & B^* B \end{bmatrix}$$

$$\left[\begin{array}{c|c} v_1^* A^* A v_1 & v_1^* A^* A \hat{V}_1 \\ \hline \hat{V}_1^* A^* A v_1 & \hat{V}_1 A^* A \hat{V}_1 \end{array}\right] = \left[\begin{array}{c|c} \sigma_1^2 & 0 \\ \hline 0 & B^* B \end{array}\right]$$

Either off-diagonal submatrix gives

$v_1^* A^* A \hat{V}_1 = 0$ (m-1 dim zero row vector)

Since $v_2 \perp v_1$ and $\{v_1;\ \text{cols}\ \hat{V}_1\}$ form basis for $\mathbb{C}^m$,

v_2 is in span cols $\hat{V}_1$, and thus $\exists\ x$ s.t. $v_2 = \hat{V}_1 x$

$v_1^* A^* A \hat{V}_1 x = 0 \cdot x$

$v_1^* A^* A v_2 = 0$ ie $Av_1 \perp Av_2$ ∎

Until a few minutes ago, I thought there was another flaw in the claim that σ_1 is uniquely determined by the condition $\sigma_1 = \|A\|_2$. But this is true, and it follows directly from the definition of the SVD

$A = U \Sigma V^*$ $\quad U, V$ unitary, Σ diagonal

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq 0 \quad \sigma_i = \Sigma_{ii}$

Given this

$\|A\|_2 = \|\cancel{U} \Sigma \cancel{V^*}\|_2 = \|\Sigma\|_2$

The 2-norm of a diagonal matrix is its largest element, and the largest elem of Σ is $\sigma_1 = \Sigma_{11}$. Therefore $\|A\| = \sigma_1$ as was to be shown.