UNH IAM 961 fall 2011 HW #1

1. Show that any linear map $\mathcal{L}: \mathbb{C}^n \to \mathbb{C}^n$ can be written as an $m \times n$ matrix.

Let $\{e_i \mid i = 1, \ldots, m\}$ be the canonical basis for $\mathbb{C}^m$

and $\{e_j \mid j = 1, \ldots, n\}$ " " $\mathbb{C}^n$

Let x be an arbitrary vector in $\mathbb{C}^n$ and let $y = \mathcal{L}x$

Expand x and y over the canonical basis sets

$$x = \sum_{j=1}^{n} x_j e_j \qquad y = \sum_{i=1}^{m} y_i e_i$$

Then

$$\sum_{k=1}^{m} y_k e_k = \mathcal{L} \sum_{j=1}^{n} x_j e_j$$

$$= \sum_{j=1}^{n} x_j \mathcal{L} e_j$$

Mult both sides by e_i^*

$$e_i^* \sum_{k=1}^{m} y_k e_k = e_i^* \sum_{j=1}^{n} x_j \mathcal{L} e_j$$

$$y_i = \sum_{j=1}^{n} x_j e_i^* \mathcal{L} e_j$$

Let $L \in \mathbb{C}^{m \times n}$ have elements $L_{ij} = e_i^* \mathcal{L} e_j$. Then

$$y_i = \sum_{j=1}^{n} L_{ij} x_j \iff y = \mathcal{L} x \quad \forall\, x \in \mathbb{C}^n$$

Thus L is a matrix representation of the map $\mathcal{L}: x \to y$

(2) Prove that $A \in \mathbb{C}^{m\times n}$ $m \geq n$ has full rank iff $Ax \neq Ay$ for all $x \neq y$ in $\mathbb{C}^n$

($\Rightarrow$) If A has full rank, then dim col space $A = n$ (by defn)

dim span cols$(A) = n$

So we have n vectors spanning an n-dim space

Therefore they must be linly indpt. (otherwise dim of their span would be less than their number)

Therefore there is nor nozero vector c s.t. $Ac = 0$

i.e. $Ac = 0 \iff c = 0$

Let $c = x - y$. Then

$$A(x-y) = 0 \iff x = y$$

$$Ax = Ay \iff x = y$$

$\therefore$ $Ax \neq Ay \iff x \neq y$ thus full rank $\Rightarrow$ $Ax \neq Ay$ for all $x \neq y$

($\Leftarrow$) If A does not have full rank, then dim col space $A < n$

Then $\exists$ lin comb of cols A $\sum_{j=1}^{n} c_j a_j = 0$ w at least one $c_j \neq 0$

ie $Ac = 0$ for som $c \neq 0$

$\Rightarrow$ $A(x+c) = Ax$

$\Rightarrow$ $\exists$ $y = x + c \neq x$ s.t. $Ax = Ay$

Thus $!$ full rank $\Rightarrow$ $\exists$ $y \neq x$ s.t. $Ax = Ay$

and full rank $\Rightarrow$ $!\exists$ $y \neq x$ s.t. $Ax = Ay$

$\therefore$ two conditions are equivalent

A has full rank $\iff$ $!\exists\, y \neq x$ s.t. $Ax = Ay$

or equiv

$Ax \neq Ay$ $\forall$ $y \neq x$

③ Prove that $\|AB\|_p \le \|A\|_p \|B\|_p$ where $\|\cdot\|_p$ is the induced matrix p-norm

$$\|AB\|_p = \sup_{x \neq 0} \frac{\|ABx\|_p}{\|x\|_p} \quad \text{by defn}$$

But $\|A\,Bx\|_p \le \|A\|_p \|Bx\|_p$ $\left(\text{by defn } \|A\|_p = \sup_{y \neq 0} \frac{\|Ay\|_p}{\|y\|_p}\right)$

So

$$\|AB\|_p \le \sup_{x \neq 0} \frac{\|A\|_p \|Bx\|_p}{\|x\|_p}$$

$$\le \|A\|_p \sup \frac{\|Bx\|_p}{\|x\|_p}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p$$

(4) Prove that $\|A\|_1 = \max_{1 \le j \le n} \|a_j\|_1$ where a_j is j^{th} col of A

$$\|A\|_1 = \sup_{\|x\|_1 = 1} \|Ax\|_1 \quad \text{by defn.}$$

Consider $\|Ax\|_1$ over x s.t. $\|x\|_1 = 1$, i.e. $\sum_{j=1}^{n} |x_j| = 1$

$$\|Ax\|_1 = \left\| \sum_{j=1}^{n} x_j a_j \right\|_1 \le \sum_{j=1}^{n} |x_j| \, \|a_j\|_1 \le \max_{1 \le j \le n} \|a_j\|_1$$

This is a weighted average of the $\|a_j\|_1$'s, so it must be less than the largest $\|a_j\|_1$

So $\sup_{\|x\|_1 = 1} \|Ax\|_1 \le \max_{1 \le j \le n} \|a_j\|_1$

But letting $x = e_j$ where j is index for which $\|a_j\|$ is maximized achieves the bound, so that

$$\|A\|_1 = \sup_{\|x\|_1 = 1} \|Ax\|_1 = \max_{1 \le j \le n} \|a_j\|_1 \quad \blacksquare$$

$u, v \in \mathbb{C}^m$

(5) Trefethen 2.6 Let $A = I + uv^*$. Show that A nonsingular $\Rightarrow A^{-1} = I + \alpha uv^*$

For what u, v is A singular? What is null A?

NLA HW#1

a) A nonsingular $\Leftrightarrow A^{-1}$ exists $\Rightarrow AA^{-1} = I$

$$(I + uv^*)(I + \alpha uv^*) = I$$

$$\cancel{I} + uv^* + \alpha uv^* + \alpha u(v^*u)v^* = \cancel{I}$$

$$uv^*(1 + \alpha + \alpha v^*u) = 0$$

either $uv^* = 0$, in which case α is arbitrary

or $\alpha = -\dfrac{1}{1 + v^*u} \in \mathbb{C}$

b) A singular $\Leftrightarrow A^{-1}$ doesn't exist. By (a) A^{-1} exists as long as $v^*u \neq -1$

so A is singular for $v^*u = -1$ (this is explicitly shown, (null $A \neq \{0\}$ when $v^*u = -1$) below)

c) null $A = \{x \in \mathbb{C}^m \mid Ax = 0\}$ assuming A is singular and thus $v^*u = -1$

$$Ax = (I + uv^*)x = 0$$

$$x + \underbrace{(v^*x)}_{\text{scalar}}u = 0$$

$\Rightarrow x = cu$ for some $c \in \mathbb{C}$

$$cu + (v^*cu)u = 0$$

$$cu + c\overbrace{(v^*u)}^{-1}u = 0$$

true $\forall c \in \mathbb{C}$

$\Rightarrow$ null $A = \{cu \mid c \in \mathbb{C}\} = \text{span } u$