Geometry of state space in plane Couette flow

P. Cvitanović and J. F. Gibson

Center for Nonlinear Science, Georgia Institute of Technology
Atlanta, GA 30332-0430, USA predrag.cvitanovic at physics.gatech.edu

Summary. A large conceptual gap separates the theory of low-dimensional chaotic dynamics from the infinite-dimensional nonlinear dynamics of turbulence. Recent advances in experimental imaging, computational methods, and dynamical systems theory suggest a way to bridge this gap in our understanding of turbulence. Recent discoveries show that recurrent coherent structures observed in wall-bounded shear flows (such as pipes and plane Couette flow) result from close passes to weakly unstable invariant solutions of the Navier-Stokes equations. These 3D, fully nonlinear solutions (equilibria, traveling waves, and periodic orbits) structure the state space of turbulent flows and provide a skeleton for analyzing their dynamics. We calculate a hierarchy of invariant solutions for plane Couette, a canonical wall-bounded shear flow. These solutions reveal organization in the flow's turbulent dynamics and can be used to predict directly from the fundamental equations physical quantities such as bulk flow rate and mean wall drag. All results and the code that generates them are disseminated through through our group's open-source CFD software and solution database `Channelflow.org` and the collaborative e-book `ChaosBook.org`.

In a seminal paper, Hopf [1] envisioned the function space of Navier-Stokes velocity fields as an infinite-dimensional state space, parameterized by viscosity, boundary conditions, and external forces, in which each $3D$ fluid velocity field is represented as a single point. As the viscosity decreases, turbulence sets in, represented by chaotic state-space trajectories. Hopf's observation that viscosity causes state-space volumes to contract under the action of dynamics led to his key conjecture: that long-term, typically observed solutions of the Navier-Stokes equations lie on finite-dimensional manifolds embedded in the infinite-dimensional state space of allowed velocity fields.

Recent experimental and theoretical advances [2] support Hopf's dynamical vision of turbulence. Space limitations prevent us from doing justice here to the fundamental 'pipes and planes' work on shear flows by Nagata, Busse, Clever, Waleffe, Holmes, Moin, Moser, Kim, Lumley, Mullin, Jiménez, Kawahara, Eckhardt, Kerswell, Tuckerman, Schmiegel, Barkley, Hof, Viswanath, and others that we build upon; we refer the reader to Refs. [3, 4, 5, 6] for an overview. The preponderance of recurrent, coherent states in wall-bounded

B. Eckhardt (ed.), *Advances in Turbulence XII*, Springer Proceedings in Physics 132,
DOI 10.1007/978-3-642-03085-7_17,

© Springer-Verlag Berlin Heidelberg 2009

shear flows suggests that their long-time dynamics lie on low-dimensional attractors and might thus be amenable to dynamical systems modeling. The qualitative success and quantitative shortcomings of low-dimensional 'Proper Orthogonal Decomposition' models [3] motivate our work [6, 7, 8, 9]: We seek to understand the dynamics of turbulence not through a low-d ODE *model*, but through a hierarchy of *exact invariant solutions* of the *fully-resolved* Navier-Stokes equations. These exact solutions turn out to be remarkably similar in appearance to coherent structures observed in both numerical simulations and experiments.

The correspondence between coherent structures and invariant solutions can be understood in terms of dynamical theory. Invariant sets (equilibria, traveling waves, periodic solutions, relative periodic solutions, their stable and unstable manifolds) *partition* state space. A trajectory within an invariant set stays within it forever, whereas a trajectory that starts outside an invariant set cannot traverse it. Thus the union of invariant sets can explain a good deal of the dynamics, and the structure imposed by invariant solutions enables systematic exploration and characterization of dynamical behaviors.

While the equilibria of a dynamical system (steady states of Navier-Stokes) do not participate in dynamics directly, their stable / unstable manifolds do shape the flow in their vicinity. The simplest time-dependent invariant solutions are periodic orbits (spatiotemporally periodic solutions of Navier-Stokes). Periodic orbits are densely embedded in the natural measure of a chaotic system. Most periodic orbits found in [9] individually capture the mean flow and Reynolds stresses of plane Couette turbulence to within a few percent. Given a hierarchical set of longer and longer such solutions, the 'trace formulas' of *periodic orbit theory* [10] should provide a systematic framework for calculating system's statistical properties (this claim is as yet untested in the context of turbulent fluid flows). Empirically, the geometry and dynamics of attractors are dominated by the *least unstable* periodic orbits. A coarse global description of dynamics is then provided by specifying the sequence of invariant solutions whose neighborhoods are visited by a chaotic trajectory, while linearization about these solutions provides highly accurate local descriptions.

In the following series of papers and our web repository, we take several steps towards realizing these goals for the case of small aspect-ratio plane Couette flow. Ref. [6] describes numerical methods for determining invariant solutions of Navier-Stokes (equilibria, periodic orbits, and their relatives; linearized stability, invariant manifolds) and a method for constructing state-space portraits of the infinite-dimensional Navier-Stokes state space. We find that projections such as Fig. 1.1 onto low-d subspaces spanned by exact equilibrium solutions reveal much about the spatiotemporal structure of turbulent dynamics. The resulting state-space portraits are dynamically invariant, intrinsic, and representation independent, and can be applied to experimental as well as numerical data.

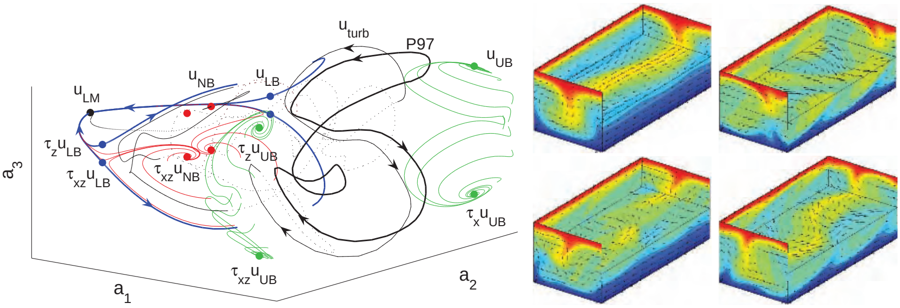

Fig. 1.1. A turbulent trajectory tracking a periodic orbit. (left panel) A turbulent trajectory (dotted line) is shown against the backdrop of unstable invariant structures, projected from 10^5 dimensions to 3. Solid dots and lines indicate equilibria and their unstable manifolds ($\mathbf{u}_{\mathrm{LM}}$ is laminar equilibrium). Key portions of the turbulent trajectory are highlighted with solid lines: close passes to a periodic orbit (thick black line) and to the unstable manifolds of several equilibria. A shorter periodic orbit –not labeled here– is also shown. (right panel) Snapshots along the periodic orbit at intervals $\Delta t = 15$, marked by open circles in the left panel, starting at the point labeled $P97$. See [6] for nomenclature and details of the state-space projection. Side-by-side animations of state-space projections and $3D$ fluid velocity fields given in `ChaosBook.org/tutorials` are particularly revealing.

Ref. [7] describes eleven equilibrium and five traveling wave solutions of plane Couette flow, most of them new, and demonstrates the robustness of these solutions under variations of Re and aspect ratios. We provide a partial classification of the isotropy subgroups of plane Couette flow and show which subgroups admit of which types of solution. Solutions are found by a novel method of generating initial guesses; namely, we start our searches from the turbulent simulation data, in contrast to more traditional continuations from / bifurcations off the known solutions.

Ref. [8] reports several heteroclinic connections amongst the equilibria solutions and shows that these connections form the backbone of transitions from multiple to single-roll states. Ref. [9] presents twenty new periodic orbit solutions to plane Couette flow and investigates how well they probe the natural measure. Lastly, our online, user-editable database of solutions at `www.channelflow.org`, aims to promote the rapid dissemination of research results within the community. The website also hosts our public domain high-level software system for plane Couette and channel flows, designed to lower the barrier to entry to research in dynamical systems and turbulence.

Together, these steps lead to a new way of thinking about coherent structures and turbulence: (a) that coherent structures are the physical images of the flow's least unstable invariant solutions, (b) that turbulent dynamics consists of walk among the set of these unstable solutions.

The long-term goals of this research program are to develop this vision into quantitative, predictive description of moderate-Re turbulence, and to use this description to control flows and explain their statistics. Open research topics include (a) Symmetry reduction of plane Couette and pipe-flow state space. (b) Construction of Poincaré sections, Poincaré maps, symbolic dynamics, and transition (Markov) graphs. (c) Extension to large and infinite aspect-ratio systems. (d) Predicting turbulent statistics from expansions over periodic orbits.

Acknowledgement. We are indebted to F. Waleffe, D. Viswanath, J. Halcrow, G. Kawahara, L.S. Tuckerman, B. Eckhardt, and D. Barkley for inspiring collaborations and/or discussions. P.C. and J.F.G. thank G. Robinson, Jr. for support. J.F.G. was partly supported by NSF grant DMS-0807574. Special thanks to the Georgia Tech Student Union which generously funded our access to the Georgia Tech Public Access Cluster Environment (GT-PACE).

References

1. Hopf, E.: A mathematical example displaying features of turbulence. Comm. Pure Appl. Math. **1** (1948) 303–322
2. Hof, B., van Doorne, C.W.H., Westerweel, J., Nieuwstadt, F.T.M., Faisst, H., Eckhardt, B., Wedin, H., Kerswell, R.R., Waleffe, F.: Experimental observation of nonlinear traveling waves in turbulent pipe flow. Science **305**(5690) (2004) 1594–1598
3. Holmes, P., Lumley, J.L., Berkooz, G.: Turbulence, coherent structures, dynamical systems and symmetry. Cambridge University Press, Cambridge (1996)
4. Panton, R.L., ed.: Self-sustaining mechanisms of wall turbulence. Computational Mechanics Publications, Southhampton (1997)
5. Robinson, S.K.: Coherent motions in the turbulent boundary layer. Annual Review of Fluid Mechanics **23** (1991) 601–639
6. Gibson, J.F., Halcrow, J., Cvitanović, P.: Visualizing the geometry of state-space in plane Couette flow. J. Fluid Mech. **611** (2008) 107–130 `arXiv:0705.3957`.
7. Halcrow, J., Gibson, J.F., Cvitanović, P.: Equilibrium and traveling-wave solutions of plane Couette flow. `arXiv:0808.3375`, J. Fluid Mech., to appear (2009)
8. Halcrow, J., Gibson, J.F., Cvitanović, P., Viswanath, D.: Heteroclinic connections in plane Couette flow. J. Fluid Mech. **621** (2009) 365–376 `arXiv:0808.1865`.
9. Gibson, J.F., Cvitanović, P.: Periodic orbits of plane Couette flow. In preparation (2009)
10. Cvitanović, P., Artuso, R., Mainieri, R., Tanner, G., Vattay, G.: Chaos: Classical and quantum. Niels Bohr Institute, Copenhagen (2009) `ChaosBook.org`.